# Complex Numbers in Geometry

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# 1 The Complex Plane

## 1.1 Definitions

I assume familiarity with most, if not all, of the following definitions. Some knowledge of linear algebra is also recommended, but not required.
Subsequently, let $i$ be the imaginary unit satisfying $i^2 = -1$. Define the set of complex numbers $\mathbb{C} = \{z \mid z = a + bi, a, b \in \mathbb{R}\}$ where $a$ is the real part of $z$ and $b$ is the imaginary part. The magnitude of a given $z = a + bi \in \mathbb{C}$ is $|z| = \sqrt{a^2 + b^2}$. The conjugate of $z = a + bi$ will be $\overline{z} = a - bi$, implying the property $|z|^2 = z \cdot \overline{z}$. As an exercise, show that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$.

Every $z \in \mathbb{C}$ can be expressed as $|z| \cdot (\cos\theta + i \sin\theta)$ for some angle $\theta \in [0, 2\pi)$, or alternatively $|z| \cdot e^{i\theta}$. This angle $\theta$ will be referred to as the argument of $z$.

Lastly, any complex number satisfying $z^n - 1 = 0$ will be referred to as an $n$ th root of unity. It can easily be verified that these numbers take the form $e^{2k\pi i/n}$ for $0 \leq k \leq n - 1$.

## 1.2 Applications to the Complex Plane

The complex plane assigns a complex number to every point in the plane such that the point $P$ with Cartesian coordinates $(a, b)$ is assigned $a + bi$. The counterpart to the $xy$-axes in the complex plane are the real and imaginary axes, respectively. From this definition, the similarity between the complex plane and the Cartesian plane should be evident. Consequently, the representation $z = |z| \cdot (cos\theta + i \sin\theta) = |z|e^{i\theta}$ corresponds to polar coordinates in the Cartesian plane.

Throughout the lecture, the lowercase letter $p$ of a point $P$ will correspond to the complex coordinate, or affix, of $P$ unless otherwise noted.

As an exercise, verify that:
- $\overline{z}$ is the reflection of $z$ over the real axis
- $\overline{\overline{z}} = z$

## 1.3 Transformations in the Complex Plane

Geometrically, in order to find the coordinates of the sum of two complex numbers, one may simply perform a vector head-to-tail addition.

Multiplication of complex numbers is slightly more interesting. For any complex numbers $z, w$, the map $z \to zw$ corresponds to a spiral similarity (composition of a dilation and rotation) about the origin. Furthermore, the magnitudes and arguments of $zw$ are determined independently in this transformation. More specifically, we have

$$|wz| = |w||z| \text{ and } \arg(w) + \arg(z) = \arg(wz).$$

Convince yourself that multiplication by a real number is equivalent to a dilation and multiplication by $e^{i\theta}$ is equivalent to a counterclockwise rotation of $\theta$.

As a side-note, any transformation $z \to \dfrac{az + b}{cz + d}$ where $a, b, c, d \in \mathbb{C}, |ac| > 0, ad - bc \neq 0$ is a Mobius transformation.

# 2 Metric Propositions

One significant advantage of complex numbers over Cartesian coordinates is that every point is assigned a single number as opposed to an ordered pair, allowing concise algebraic expressions of geometric concepts. From these fundamental propositions, the advantages of complex numbers in various configurations should manifest themselves.

**Proposition 2.1**. (Angle Between Two Lines).
Let the angle formed by two lines $AB, CD$ in the clockwise direction from $AB$ to $CD$ be $\theta = \measuredangle(AB, CD)$. Then

$$\frac{a-b}{|a-b|} = e^{i\theta}\frac{c-d}{|c-d|}.$$

*Proof.* Translate the segments such that $B, D$ coincide with the origin, with $A' = a-b, C' = c-d$. Then $\dfrac{\frac{a-b}{|a-b|}}{\frac{c-d}{|c-d|}} = \dfrac{r_1 e^{i\theta_1}/r_1}{r_2 e^{i\theta_2}/r_2} = e^{i\theta_1}/e^{i\theta_2} = e^{i(\theta_1-\theta_2)} = e^{i\theta}$.

This result serves as the basis for many useful corollaries.

**Corollary 2.2.** (Alternate form of 1)
Squaring both sides of Proposition 2.1 yields the more applicable form

$$\frac{a-b}{\overline{a}-\overline{b}} = e^{2i\theta}\frac{c-d}{\overline{c}-\overline{d}}.$$

**Corollary 2.3.** (Collinearity)
Points $A, B, C$ are collinear iff

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{c-b}{\overline{c}-\overline{b}}.$$

*Proof.* It remains to show that the angle between $AB, CB$ is 0, which follows from letting $\theta = 0$ in Corollary 2.2. Note that this can also be rearranged to $\dfrac{b-a}{b-c} = \overline{\left(\dfrac{b-a}{b-c}\right)}$.

**Corollary 2.4.** (Equation of a Line)
From Corollary 2.3, letting c be a variable point gives the equation of a line.

**Corollary 2.5.** (Perpendicularity)
Segments $AB, CD$ are perpendicular iff

$$\frac{a-b}{\overline{a}-\overline{b}} = -\frac{c-d}{\overline{c}-\overline{d}}.$$

*Proof.* Let $\theta = \pi/2$ in Corollary 2.2.

**Proposition 2.6.** (Directly Similar Triangles)
$\triangle ABC \sim \triangle DEF$ and $ABC, DEF$ are similarly oriented iff

$$\frac{a-b}{a-c} = \frac{d-e}{d-f}.$$

*Sketch.* Consider the magnitudes of both sides of the equation to get $\dfrac{AB}{AC} = \dfrac{DE}{DF}$. Now dividing this with the given condition and applications of Proposition 2.1, we get $\angle BAC = \angle EDF$.

**Corollary 2.7** (Spiral Similarity)
The center of spiral similarity taking $\overline{AB} \to \overline{CD}$ is given by $\dfrac{ad-bc}{a+d-b-c}$.

*Proof.* By Proposition 2.5, it suffices to solve $\dfrac{p-a}{p-b} = \dfrac{p-c}{p-d}$ for $p$.

**Proposition 2.8** (Cyclicity)
Points $A, B, C, D$ are concyclic iff

$$\frac{(a-c)(b-d)}{(\overline{a}-\overline{c})(\overline{b}-\overline{d})} = \frac{(a-d)(b-c)}{(\overline{a}-\overline{d})(\overline{b}-\overline{c})}.$$

*Sketch.* Rewrite $\angle ACB = \angle ADB$ with Proposition 2.1.

**Proposition 2.9.** The area of triangle $ABC$ is

$$\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix} = a(\overline{b}-\overline{c}) - \overline{a}(b-c) + (b\overline{c} - \overline{b}c).$$

*Sketch.* Expand with Shoelace formula and Cartesian coordinates; details are left as an exercise to the reader.

**Proposition 2.10.** (Reflection About a Line)
The reflection of $P$ with respect to line $AB$, denoted by $z$, satisfies

$$z = \frac{(a-b)\overline{p} + \overline{a}b - a\overline{b}}{\overline{a}-\overline{b}}.$$

*Sketch.* The proof of this uses the fact that linear transformations preserve reflections, so taking $z \to \dfrac{z-a}{b-a}$ and noting that $\overline{AB}$ is mapped to the real axis (more specifically, the line segment containing 0 and 1) reaches the conclusion.

**Proposition 2.11.** (Circumcenter Formula)
The circumcenter of $\triangle ABC$, denoted by $x$, satisfies $x = \dfrac{\begin{vmatrix} a & a\overline{a} & 1 \\ b & b\overline{b} & 1 \\ c & c\overline{c} & 1 \end{vmatrix}}{\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}.$

*Sketch.* Consider the radius $R$ of the circumcircle, Then $|x-a|^2 = |x-b|^2 = |x-c|^2 = r^2$ so expanding we get a system of equations in $x, \overline{x}, r^2 - |x|^2$ which we can solve with Cramer's rule.

# 3 The Unit Circle

One of the fundamental properties of the complex plane that make enormous computations viable is the fact that for any point $p$ on the unit circle, $\overline{p} = \dfrac{1}{p}$. Since any circle can be mapped to the unit circle via a composition of translation and dilation, it is often useful to let a cumbersome circle to be the unit circle.

**Proposition 3.1** (Equation of a Chord)
If $AB$ is a chord of the unit circle, the equation of line $AB$ is given by

$$z = a + b - ab\overline{z}.$$

*Proof.* From Corollary 2.4, $\dfrac{z-a}{\overline{z}-\overline{a}} = \dfrac{a-b}{\overline{a}-\overline{b}} \iff \dfrac{z-a}{\overline{z}-\frac{1}{a}} = \dfrac{a-b}{\frac{1}{a}-\overline{b}} = -ab$ and upon expanding we get the desired result. Remark on the simplicity of this equation, possible only by this property of the unit circle.

**Corollary 3.2** (Chord Intersection)
Let $AB, CD$ be two chords on the unit circle. If $P = AB \cap CD$ then

$$p = \frac{ab(c+d) - cd(a+b)}{ab - cd}$$

if $ab - cd \neq 0$.
*Sketch.* From Proposition 3.1, we know that $p$ is a solution to both

$$\begin{cases} p = a + b - ab\overline{p} \\ p = c + d - cd\overline{p}. \end{cases}$$

Subtracting these two equations and solving for $\overline{p}$, we consequently get $p$ after some simplification.

**Corollary 3.3** (Tangent Intersection)
If the tangents to the unit circle at $A, B$ intersect at $P$, then

$$p = \frac{2ab}{a+b}.$$

*Proof.* Note that a tangent is simply a degenerate chord; substituting $AA, BB$ into Corollary 3.2 we get the desired.

Another nice property of unit circles regards fundamental triangle centers of triangles inscribed in the unit circle.

**Proposition 3.4** (Orthocenter, Centroid, Nine-Point Center)
For any triangle $ABC$ inscribed in the unit circle, its orthocenter, centroid, and nine-point center is given by

$$a + b + c, \frac{a+b+c}{3}, \frac{a+b+c}{2}$$

respectively.
*Proof.* Note that from knowledge of Cartesian coordinates, the centroid is $\frac{a+b+c}{3}$ regardless of origin. The rest follows from the Euler Line.

**Proposition 3.5** (Incenter, Excenters, and Midpoints of Arcs)
Let $ABC$ be a triangle inscribed in the unit circle, and let the complex coordinates of $A, B, C$ be $a^2, b^2, c^2$ for complex numbers $a, b, c$. Let $A_1, B_1, C_1$ have coordinates $-bc, -ca, -ab$, $I$ have coordinate $-ab - bc - ca$, and finally $I_a, I_b, I_c$ have coordinates $ca + ab - bc, ab + bc - ca, bc + ca - ab$ respectively. Then $A_1, B_1, C_1$ are midpoints of arcs $BC, CA, AB$, $I$ is the incenter of $ABC$, and $I_a, I_b, I_c$ are the $A, B, C$-excenters.

*Proof.* Since $|-bc| = |b| * |c| = 1$ it lies on the unit circle. Furthermore, by Proposition 2.1 we have $e^{2i\angle A_1AB} = \frac{\frac{b^2-a^2}{\overline{b^2}-\overline{a^2}}}{\frac{-bc-a62}{-\overline{bc}-\overline{a}^2}} = \frac{-a^2b^2}{a^2bc} = -\frac{b}{c}$. Similarly $e^{2i\angle A_1AC} = -\frac{c}{b}$ so $A_1$ is on the $A$-angle bisector, implying that it is indeed the midpoint of arc. We obtain similar results for $b_1, c_1$. To prove that $I$ has coordinate $-ab - bc - ca$, note that $I$ is the orthocenter of $A_1B_1C_1$ and apply Proposition 3.4. The proof for the excenters is left to the reader as an exercise.

**Proposition 3.6** (Regular Polygons and Roots of Unity)
Let $P_1, P_2, \cdots, P_n$ in the complex plane satisfy $P_k = \omega^k$, where $\omega = e^{2\pi i/n}$. Then $P_1P_2\cdots P_n$ is a regular $n$-gon.

*Proof.* Evidently $|p_k| = 1$, so all of the points lie on the unit circle. Furthermore, it can easily be confirmed that $|p_{k+1} - p_k|$ is constant for $1 \leq k \leq n, p_{n+1} = p_1$ and similarly that $\angle P_kP_{k+1}P_{k+2}$ is constant, done.

# 4 Some Instructive Examples

When bashing, it is helpful to let a prominent circle in the diagram be the unit circle, or otherwise configure the diagram such that the coordinates have nice values. However, it should be noted that when doing so, the diagram should be general enough to encompass all possible cases.

**Problem 4.1** Let $ABC$ be a triangle with circumcenter $O$. Suppose $D$ and $E$ lie on $AB$ and $AC$, respectively, such that $O$ lies on $DE$. Let $M$ and $N$ be the midpoints of $CD$ and $BE$, respectively. Prove that $\angle MON = \angle BAC$. *(Iran 2004 / WOOT)*

*Proof.* WLOG let $\omega$, the circumcircle of $ABC$, be the unit circle, and further let $DE$ coincide with the real axis. (Convince yourself that these assumptions are sufficiently general.) Then from the definition of point $D$, $A, B, D$ are collinear $\implies \frac{a-b}{\overline{a}-\overline{b}} = \frac{a-d}{\overline{a}-\overline{d}}$. But since $D$ lies on the real axis, $d = \overline{d}$ and solving for $d$ yields

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{a-d}{\overline{a}-\overline{d}}$$

$$(a-b)(\overline{a}-d) = (a-d)(\overline{a}-\overline{b})$$

$$a\overline{a} - ad - \overline{a}b + bd = a\overline{a} - a\overline{b} - \overline{a}d + \overline{b}d$$

$$d(b - a + \overline{a} - \overline{b}) = -a\overline{b} + \overline{a}b$$

$$d = \frac{-a\overline{b} + b\overline{a}}{-a + \overline{a} + b - \overline{b}} = \frac{-a^2 + b^2}{-a^2b + b + ab^2 - a} = \frac{(b-a)(b+a)}{(b-a)(ab+1)} = \frac{b+a}{ab+1}.$$

Hence, the midpoint of $CD$ is simply $m = \frac{d+c}{2} = \frac{abc+c+b+a}{2(ab+1)}$, and similarly $n = \frac{abc+a+b+c}{2(ac+1)}$

Thus $e^{2i\measuredangle MON} = \left(\frac{m}{n}\right) \cdot \left(\frac{\overline{n}}{\overline{m}}\right) = \frac{\frac{abc+c+b+a}{2(ab+1)}}{\frac{abc+c+b+a}{2(ac+1)}} \cdot \frac{\frac{\overline{abc+a+b+c}}{2(\overline{ac}+1)}}{\frac{\overline{abc+a+b+c}}{2(\overline{ab}+1)}} = \frac{ac+1}{ab+1} \cdot \frac{\overline{ab}+1}{\overline{ac}+1} = \frac{ac+1}{ab+1} \cdot \frac{c+abc}{b+abc} = \frac{c}{b}.$

But $e^{2i\measuredangle BAC} = \frac{a-c}{\overline{a}-\overline{c}} \cdot \frac{\overline{a}-\overline{b}}{a-b} = \frac{a^2c - ac^2}{c-a} \cdot \frac{b-a}{a^2b - ab^2} = \frac{-ac}{-ab} = \frac{c}{b}$, so we may conclude.

**Problem 4.2** Let $ABC$ be an acute triangle with $AB, AC, BC$. Denote by $O, H$ the circumcenter and orthocenter of $\triangle ABC$. Suppose that the circumcircle of $\triangle AHC$ intersects $AB$ again at $M$ and the circumcircle of $\triangle AHB$ intersects $AC$ again at $n$. Prove that the circumcenter of $\triangle MNH$ lies on line $OH$. *(APMO 2010)*

*Proof.* This problem illustrates the importance of synthetic observations when complex bashing. Once we assume that $\omega$ is the unit circle, it is very difficult to obtain coordinates for points $M, N$ without massive computations. Hence, we need some sort of synthetic insight.

Let $D$ be the foot of the $C$-altitude, and $P$ be the circumcenter of $\triangle MNH$. Easily obtain the coordinates of $D$ by reflecting $C$ across $AB$, and taking the midpoint of $CC'$.

Note that $\angle CMB = \pi - \angle AMB = \pi - \angle AHB = \angle C$, so $BMC$ is isosceles, and it thus follows that $D$ is the midpoint of $MC$. Now $d = \frac{a+b+c-ab\overline{c}}{2} \implies m = a + c - ab\overline{c}$, and similarly we get $n = a+b-a\overline{b}c$. Note that $O = 0, H = a+b+c$ so a translation of $-a-b-c$ will still yield the circumcenter of $MNH$ on line $OH$. Then $m' = \frac{-b(a+c)}{c}$, and similarly $n' = \frac{-c(a+b)}{b}, h' = 0$. Then by the circumcenter formula,

$$p' = \frac{\left(\frac{b(a+c)}{c}\right)\left(\frac{c(a+b)}{b}\right)\left(\frac{a+b}{ac} - \frac{a+c}{ab}\right)}{\left(\frac{a+c}{ab}\right)\left(\frac{c(a+b)}{b}\right) - \left(\frac{a+b}{ac}\right)\left(\frac{b(a+c)}{c}\right)}$$

$$p' = \frac{(a+c)(a+b)\left(\frac{a+b}{ac} - \frac{a+c}{ab}\right)}{(a+c)(a+b)\left(\frac{c}{ab^2} - \frac{b}{ac^2}\right)} = -\frac{bc(a+b+c)}{b^2+bc+c^2}.$$

It suffices to show $p', o, h$ collinear, but this follows from

$$\frac{p'-o}{\overline{p'}-\overline{o}} = \frac{-\frac{bc(a+b+c)}{b^2+bc+c^2}}{-\frac{bc+ca+ab}{a(b^2+bc+c^2)}} = \frac{abc(a+b+c)}{ab+bc+ca} = \frac{h-o}{\overline{h}-\overline{o}},$$

and we may conclude.

**Problem 4.3** In $\triangle ABC$, let $O$ be the circumcenter of $(ABC)$, $A_1$ to be the antipode of $A$ WRT $(ABC)$, $A_2$ to be the reflection of $O$ across $AB$. Let $O_A$ be the circumcenter of $A_1A_2O$. Define $O_B, O_C$ similarly. Prove that $O_A, O_B, O_C$ are collinear. *(Lemmas in Olympiad Geometry)*

*Proof.* Evidently $O_A \in BC$ and similar, since the perpendicular bisector of $OA_2$ is $BC$ by definition of reflection. Now it suffices to show that $\frac{BO_A}{CO_A} \cdot \frac{CO_B}{AO_B} \cdot \frac{AO_C}{BO_C} = 1$ by Menelaus. To do so, we use complex numbers. Let $(ABC)$ be the unit circle so that $A_1 = -a, A_2 = b+c$. Then $O_A = \frac{a_1a_2}{\overline{a_1}a_2 - a_1\overline{a_2}} = \frac{-a(b+c)(-\overline{a}-\overline{b}-\overline{c})}{\overline{a}(b+c) - a(\overline{b+c})}$.

Now $\frac{|BO_A|}{|CO_A|} = \left|\frac{\frac{a+b+c+a\overline{b}+\overline{a}b^2+\overline{a}bc}{\overline{a}(b+c)-a(\overline{b+c})}}{\frac{a+b+c+ab\overline{c}+\overline{a}c^2+\overline{a}bc}{\overline{a}(b+c)-a(\overline{b+c})}}\right| = \left|\frac{a+b+c+a\overline{b}+\overline{a}b^2+\overline{a}bc}{a+b+c+ab\overline{c}+\overline{a}c^2+\overline{a}bc}\right| = \left|\frac{a+b+c+a\overline{b}+\overline{a}b^2+\overline{a}bc}{a+b+c+ab\overline{c}+\overline{a}c^2+\overline{a}bc}\right| \cdot \frac{|abc|}{|abc|} = \frac{|c|}{|b|} \cdot \frac{a^2b+ab^2+abc+a^2+b^3+b^2c}{a^2c+abc+ac^2+a^2b+c^3+bc^2} = \frac{|c|}{|b|} \cdot \frac{\cancel{|b+c|} \cdot |a^2+ab+b^2|}{\cancel{|b+c|} \cdot |a^2+ac+c^2|} = \frac{|c| \cdot |a^2+ab+b^2|}{|b| \cdot |a^2+ac+c^2|}$, at which point it is clear that multiplying cyclically will give the desired result.

**Problem 4.4** In $\triangle ABC$ with incenter $I$, the incircle is tangent to $CA, AB$ at $E, F$. The reflection of $E, F$ across $I$ are $G, H$. Let $Q = BC \cap GH$, and let $M$ be the midpoint of $\overline{BC}$. Prove that $IQ \perp IM$.

*Proof.* Let the incircle be tangent to $BC$ at $D$, and let the incircle be the unit circle. This gives nice coordinates from chord intersection formulas, and moreover we get $g = -e$ and $h = -f$. Then $A = EE \cap FF$ so $a = \frac{2ef}{e+f}$ and similar coordinates can be derived for $B$ and $C$. Next

$$Q = BC \cap GH = DD \cap GH \implies \frac{d^2(g+h) - gh(d+d)}{d^2 - gh} = \frac{-d^2(e+f) - 2def}{d^2 - ef},$$

$$M \implies \frac{\left(\frac{2fd}{f+d}\right) + \left(\frac{2de}{d+e}\right)}{2} = \frac{fd^2 + 2def + ed^2}{(d+f)(d+e)}.$$

By the perpendicularity condition, it suffices to show $\frac{q}{\overline{q}} = -\frac{m}{\overline{m}} \iff \frac{q}{m} = -\overline{\left(\frac{q}{m}\right)}$.

But

$$\frac{q}{m} = \frac{-d\left(\frac{de+df+2ef}{d^2-ef}\right)}{d\left(\frac{df+2ef+de}{(d+f)(d+e)}\right)} = -\frac{d^2+ed+df+ef}{d^2-ef} = \frac{\frac{1}{d^2}+\frac{1}{ed}+\frac{1}{df}+\frac{1}{ef}}{\frac{1}{d^2}-\frac{1}{ef}} = -\overline{\left(\frac{q}{m}\right)},$$

so we may conclude.

## 5 Practice Problems

**Problem 5.1** A quadrilateral $ABCD$ is inscribed in unit circle.If $P = AB \cap CD$ and $Q = AD \cap BC$, then prove that

$$p \cdot \overline{q} + q \cdot \overline{p} = 2.$$

**Problem 5.2** Let $ABC$ be a triangle with incricle $\Gamma$ and let $D, E, F$ be the tangency points of $\Gamma$ with $BC, CA, AB$, respectively. Let $K$ be the orthocenter of triangle $DEF$ and let $\Gamma_A, \Gamma_B, \Gamma_C$ be circles centered at $A, B, C$ with radii $AD, BE, CF$, respectively. Prove that $K$ is the radical center of $\Gamma_A, \Gamma_B, \Gamma_C$. *(AMSP Geo 3)*

**Problem 5.3** Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle $ABC$ and a point $D$ such that $DB$ and $DC$ are tangent to the circumcircle of $ABC$. Let $B'$ be the reflection of $B$ over $AC$ and $C'$ be the reflection of $C$ over $AB$. If $O$ is the circumcenter of $DB'C'$, help Oscar prove that $AO$ is perpendicular to $BC$. *(ELMO, 2016)*

**Problem 5.4** Quadrilateral $APBQ$ is inscribed in circle $\omega$ with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let $X$ be a variable point on segment $\overline{PQ}$. Line $AX$ meets $\omega$ again at $S$ (other than $A$). Point $T$ lies on arc $AQB$ of $\omega$ such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let $M$ denote the midpoint of chord $\overline{ST}$. As $X$ varies on segment $\overline{PQ}$, show that $M$ moves along a circle. *(USAJMO, 2015)*

**Problem 5.5** Let $P$ be a point in the plane of triangle $ABC$, and $\gamma$ a line passing through $P$. Let $A'$, $B'$, $C'$ be the points where the reflections of lines $PA$, $PB$, $PC$ with respect to $\gamma$ intersect lines $BC$, $AC$, $AB$, respectively. Prove that $A'$, $B'$, $C'$ are collinear. *(USAMO, 2012)*

**Problem 5.6** Let $ABC$ be a triangle and let $P$ be a point on its circumcircle. Let $X, Y, Z$ be the reflections of $P$ over lines $BC, CA, AB$ respectively. Then $X, Y, Z, H$ are collinear where $H$ is the orthocenter of $\triangle ABC$. *(Steiner Line)*

**Problem 5.7** Prove that the incircle of $\triangle ABC$ is tangent to the nine-point circle of $\triangle ABC$. *(Feuerbach)*

**Problem 5.8** Let $ABCD$ be a cyclic quadrilateral centered at $O$ and let $E = AC \cap BD$, $F = AB \cap CD$, and $G = AD \cap BC$. Prove that $O$ is the orthocenter of $\triangle EFG$. *(Brokard)*