# Trigonometry

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* You should attempt to prove every theorem or identity provided. Do not simply memorize the formulas becuase there is a lot to absorb. If you successfully come up with the proof, you will be able to derive the formulas whenever.

## 1 Basic Definitions

You are most likely already familiar with these definitions (at least I hope so). $\sin x = \frac{opposite}{hypotenuse}$, $\cos x = \frac{adjacent}{hypotenuse}$, and $\tan x = \frac{opposite}{adjacent} = \frac{\sin x}{\cos x}$. These definitions only apply in a right triangle. There are also functions defined as the reciprocal functions of the aforementioned. $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\cot x = \frac{1}{\tan x}$.

## 2 Special Right Triangles

In certain right triangles the trigonometric functions of the angles come out to very nice numbers. They go as follows: $\sin 30° = \frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, etc... (You should know these as well)

## 3 Inverse Functions

You learned what sin, cos, and tan were. So now lets move on to their inverse functions. $\sin^{-1} x$ or $\arcsin x$ means the angle whose sin is equivalent to $x$. The same goes for cos and tan.

## 4 Trigonometric Identities

Firstly you should be familiar with the following: $\cos(180 - x) = -\cos x$, $\sin(180 - x) = \sin x$, and $\sin(90 - x) = \cos x$. Now let's talk about some more complex identities. By constructing figures and performing some algebra, we can come up with the following identities: $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, and $\cot^2 x + 1 = \csc^2 x$. These are known as the *Pythagorean Identities*. Then

there are *addition* and *subtraction* identities.
$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$

Some other identities derived from the addition identities are:
$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$ (Use the pythagorean identities to come up with the 2 alternative forms of this.)

*For further exploitation of the angle formulas, such as $\sin 3x$, simply use the addition formulas with $2x$ and $x$ as angles.

# 5 Law of Cosines

In $\triangle ABC$, let the sides opposite to angles $A$, $B$, and $C$ be $a$, $b$, and $c$ respectively. Then the Law of Cosines states: $a^2 + b^2 - 2ab\cos C = c^2$. Where the variables are interchangeable.

# 6 The Extended Law of Sines

Considering the same triangle as in the Law of Cosines, we have the following equivalance: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circumcircle of $\triangle ABC$.

# 7 Other Useful Theorems

(Remember to prove)

1. $\dfrac{abc}{4R} = [ABC]$

2. $\frac{1}{2}ab\sin C = [ABC]$

3. $a = b\cos C + c\cos B$

4. $a(\sin B - \sin C) + b(\sin A - \sin C) + c(\sin A - \sin B) = 0$

# 8 Complex Number Representations

Any number you can think of can be expressed as $r(\cos x + i \sin x)$, or in shorthand $r(cisx)$. Expressing numbers in this way leads to many uses because the complex number $i$ has a lot of interesting properties.
**De Moivre's Theorem**
$[r(\cos x + i \sin x)]^n = r^n[\cos(xn) + i \sin(xn)]$

# 9 Problems

1. If $\sec x - \tan x = 2$, compute $\sec x + \tan x$.

2. Compute
   (a) $\sin \frac{\pi}{12}$, $\cos \frac{\pi}{12}$, and $\tan \frac{\pi}{12}$.

   (b) $\cos^4 \frac{\pi}{24} - \sin^4 \frac{\pi}{24}$

   (c) $\cos 36° - \cos 72°$

   (d) $\sin 10° \sin 50° \sin 70°$

3. Simplify the expression.
   $\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$.

4. Prove that $\csc \frac{180}{7}^\circ = \csc \frac{360}{7}^\circ + \csc \frac{540}{7}^\circ$.

5. In $\triangle ABC$, $\angle C = 3\angle A$. If $BC = 27$ and $AB = 48$, find $AC$.

6. Show that $\cot 10° - 4\cos 10° = \sqrt{3}$.

7. Compute $10\cot(\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21)$.

8. Find the radius of the inscribed circle of $\triangle ABC$ if $\cos^2 \frac{A}{2} = \frac{b+c}{2c} = \frac{9}{10}$, $c = 5$.

9. In $\triangle ABC$, $BC = a$, $CA = b$, and $AB = c$. If $9a^2 + 9b^2 - 19c^2 = 0$, find the value of $\dfrac{\cot C}{\cot A + \cot B}$.

10. In $\triangle ABC$, the length of the sides opposite to angles A, B, and C are $a$, $b$, and $c$ respectively. The area of triangle $ABC$ is equal to $a^2 + b^2 - c^2$. If $\angle C$ is acute, compute the numerical value of its secant, where $a$, $b$, and $c$ are positive real numbers.

11. Isosceles $\triangle ABC$ has a right angle at $C$. Point $P$ is inside the triangle such that $PA = 11$, $PB = 7$, and $PC = 6$. Legs $AC$ and $BC$ have length $s = \sqrt{a + b\sqrt{2}}$, where $a$ and $b$ are positive integers. What is $a + b$?

12. A circle centered at $O$ has radius 1 and contains the point $A$. Segment $AB$ is tangent to the circle at $A$ and $\angle AOB = \theta$. If point $C$ lies on $OA$ and $BC$ bisects $\angle ABO$, then express $OC$ in terms of $\theta$.

13. Find the value of $\beta + \frac{\alpha}{2}$ if $3\sin\alpha = 2\sin\beta$ and $3\cos\alpha + 2\cos\beta = 3$. Where both $\alpha$ and $\beta$ are acute angles.

14. In a triangle with sides of lengths $a$, $b$, and $c$, $(a+b+c)(a+b-c) = 3ab$. If $\sin^2 A = \sin^2 B + \sin^2 C$, then the three interior angles of $\triangle ABC$ are?

15. If $z + \dfrac{1}{z} = 2\cos 3$, then compute $\lfloor z^{2006} + \dfrac{1}{z^{2006}} \rfloor$.

16. In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let $M$ be the midpoint of $CA$, and let $D$ be the point on $CA$ such that $BD$ bisects angle $ABC$. Let $F$ be the point on $BC$ such that $DF \perp BD$. Suppose that $DF$ meets $BM$ at $E$. The ratio $DE : EF$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

17. In $\triangle ABC$, if $\sin A - 2\sin B\cos C = 0$, then what must be true about $\triangle ABC$? (i.e it is a right triangle, it is equilateral, etc...)

18. $BC$ is a diameter of circle $O$. $AC$ is tangent to $O$ at $C$. $AC = BC$. Draw $CE$ perpendicular to $AO$ to meet $AO$ at $F$, and $AB$ at $E$. Show that $AE = 2BE$.

19. In triangle $ABC$, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. Then $\angle C$ in degrees is?

20. In a triangle with sides of length $a$, $b$, and $c$, $(a+b+c)(a+b-c) = 3ab$. The measure of the angle opposite the side of length $c$ is?