Synthetic solution by using equilateral triangles

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There are some geometry problems in which observation tells you to use angle chasing but somehow you cannot manage to find an answer.

1 Example

We have a triangle $ABC.D$ is a point on side AC and E is a point on side BC such that $\angle CAE = 20°, \angle EAB = 60°, \angle DBA = 50°, \angle DBC = 30°$. Find$\angle DEA$

Note that after some angle chasing (in which you have to do on your own), you will not be able to find $\angle DEA$ although you know that it have some particular value since D, E are fixed points with respect to ABC.

2 Introduction

Today, I am going to show you how equilateral triangles can be applied effectively in questions that seem to not work under angle-chase.

1) Firstly, I will talk about why they are useful. Equilateral triangles have two properties that are equivalent: All angles are $60°$ and all sides have the same length. If we known one such condition exists, then we can use the other one.

2) Now, I display some of my methods that I use to construct equilateral triangles to help me.
i) Find a $60°$ angle. Contruct a equilateral triangle from it. For example, If we have a $40°, 60°, 80°$, then we dissect the $80°$ angle into a $60°$ and a $20°$ triangle.
ii) If you have a $30°$ angle, flip it over one of the adjacent sides and you will get case i)
iii) If you do not have a $30°$ nor a $60°$ angle, try to make one.

3 Solving our example

We have a triangle $ABC.D$ is a point on side AC and E is a point on side BC such that $\angle CAE = 20°, \angle EAB = 60°, \angle DBA = 50°, \angle DBC = 30°$. Find$\angle DEA$.

Solution: Construct the point F on BC such that $\angle FAB = 20°$.

Then $\angle AFB = 80° = \angle ABF$ so $AF = AB$. Note that $AD = AB = AF$ and $\angle DAF = 60°$ so $\triangle DAF$ is equilateral.

$\angle EAF = \angle EAB - \angle FAB = 40° = \angle AEB$ so $DF = AF = EF$ so $\angle DEF = \frac{180° - 40°}{2} = 70°$ so $\angle DEA = 30°$.

4 Practice

1) In triangle ABC, $\angle A = 40°, \angle B = 60°$. The bisector of $\angle A$ cuts BC at D. F is a point on AB such tha t $\angle ADF = 30°$. What is the measure of $\angle DFC$?.

2) P is a point inside triangle ABC such that$\angle PBC = 30°$, $\angle PBA = 8°$ and $\angle PAB = \angle PAC = 22°$. Find $\angle APC$