# Simon's Favorite Factoring Trick

Eugenis

May 31, 2015

## 1 Introduction

Simon's Favorite Factoring Trick (SFFT) is a direct application of grouping that is used to solve many problems. We will begin with some trivial problems that use SFFT directly, into some more advanced Olympiad problems. I will leave the Olympiad problems for the reader to solve. Instead of simply explaining SFFT in generally terms, I will introduce the technique through an assortment of problems, varying in difficulty.

### 1.1 Exercises

**Exercise 1.** Factor $xy + x + y$ as the product of two binomials $\pm$ a constant.
**Solution 1.** Let us first pull out a $x$ term out of the first two terms. This factors as

$$x(y + 1) + y$$

Note that we can factor by grouping if and only if we had another $y + 1$ term. By wishful thinking, we find this is indeed achievable. By adding and subtracting a positive 1 to the expression, we can factor by grouping.

$$x(y + 1) + 1(y + 1) - 1 = (x + 1)(y + 1) - 1$$

Our desired result is $\boxed{(x + 1)(y + 1) - 1}$.

**Exercise 2.** Factor $4xy + 6x + 10y$ by Simon's Favorite Factoring Trick (SFFT).
**Solution 2.** As done previously, let us pull out a $x$ term out of the first two terms. This factors as

$$x(4y + 6) + 10y$$

However note that it is not currently possible to create another $2y + 1$ term. Clearly the problem is where the $xy$ coefficient was not equal to 1. This is extremely valuable advice when trying to use SFFT; we want the coefficient of the $xy$ term to be equal to 1. Using that advice, let us redo the problem. We need to pull out a factor of 4 out of all the terms. This yields

$$4\left(xy + \frac{3}{2}x + \frac{5}{2}y\right)$$

Using wishful thinking as seen previously, this factors as

$$4\left(\left(x + \frac{5}{2}\right)\left(y + \frac{3}{2}\right) - \frac{15}{4}\right)$$

Cleaning the problem up by distributing yields

$$\boxed{(2x + 5)(2y + 3) - 15}.$$

## 2 Challenge Problems

**Problem 1** [2007 AMC 12] How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

**Problem 2** [1998 AIME] An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where $m, n$, and $p$ are integers, and $m \leq n \leq p$. What is the largest possible value of $p$?

**Problem 3** [2005 BMO] The integer $N$ is positive. There are exactly 2005 ordered pairs $(x, y)$ of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$

Prove that $N$ is a perfect square.

**Problem 4** [2003 JBMO] Let $n$ be a positive integer. A number $A$ consists of $2n$ digits, each of which is 4; and a number $B$ consists of $n$ digits, each of which is 8. Prove that $A + 2B + 4$ is a perfect square.

**Problem 5** [2000 AIME] The system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0 \end{aligned}$$

has two solutions $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. Find $y_1 + y_2$.