

Pythagorean triples in elementary number theory

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1 Introduction

Pythagorean triples are integer solutions to the well known Pythagorean theorem, $a^2 + b^2 = c^2$. Although it is a geometrical theorem, the part that the sum of two squares is another square is a number theory problem. In this article, we will develop a formula for a subset of these triples, and then find a formula for all Pythagorean triples. There will be challenge questions at the end of most both sections, going over topics similar to that of the section.

2 Leg and hypotenuse differ by 1

In other words, this section is about $c - b = 1$, where $b > a$. First to develop a formula for these triples, we look at examples of them. here are the first few examples of them, (3,4,5) (5,12,13), (7,24,25), (9,40,41). First, we look at parity. We always have (odd, even, odd). This is always true, which you will prove as an exercise. To start giving these numbers variables, call the even leg $2x$, and the hypotenuse $2x + 1$. Also the other leg will be y for now, as we do not know what it is in terms of x yet. Using the Pythagorean theorem, we find what y is in terms of x .

$$y^2 + (2x)^2 = (2x + 1)^2$$

$$y^2 = 4x + 1$$

$$y = \sqrt{4x + 1}$$

Now our triple is $(\sqrt{4x + 1}, 2x, 2x + 1)$. However, $\sqrt{4x + 1}$ is not always an integer. We need $4x + 1$ to equal an odd perfect square, or $4x + 1 = (2c + 1)^2$. We now solve for x .

$$4x + 1 = 4c^2 + 4c + 1$$

$$x = c^2 + c = c(c + 1)$$

Plugging this into our triple, we now get $(2c + 1, 2c^2 + 2c, 2c^2 + 2c + 1)$. Just to make sure our formula works, we plug this into the Pythagorean theorem.

$$(2c+1)^2 + (2c^2+2c)^2 = 4c^2+4c+1+4c^4+8c^3+4c^2 = 4c^4+8c^3+8c^2+4c+1 = (2c^2+2c+1)^2$$

The most interesting part of this formula is that the smallest leg squared is equal to the sum of the larger leg and hypotenuse. This formula also shows that any odd number greater than 1 is in a Pythagorean triple because of the $2c + 1$ term. combining these two, we can easily make a triple with any odd number.

For example, let's say one of our legs is 13. The square of 13, 169, is the sum of the leg and hypotenuse. Also, in this formula, we have the hypotenuse is 1 larger than the large leg. Call the large leg x , we have $x + x + 1 = 169$ or $x = 84$. This means our triple is (13,84,85). Remember, we can only do this when we want our leg and hypotenuse to be 1 apart.

2.1 Challenge Problems

1. Prove that if the hypotenuse and large leg of a right triangle differ by 1, the small leg is odd, large leg is even, and hypotenuse is odd.
2. With our formula for these kind of triples, find a recurrence formula for the large leg. *Hint, Let the formula for the large leg be $f(c)$, Find out what y is in terms of c in $f(c) = f(c - 1) + y^*$
3. Find the triple with smallest leg 23.

3 General case

Now that we have some intuition, let's try the harder problem, finding the general case. The key part is making observations. Let's look at some Pythagorean triples to find some pattern. Without loss of generalization, say in the triple (a,b,c) we have a and c odd with b even. We notice $a - c$ is always double a perfect square. Also, we notice that b is divisible by the square root of that perfect square. Now our triple is $(x - y^2, 2zy, x + y^2)$ for some natural numbers z, x and y . Now we use the Pythagorean theorem.

$$\begin{aligned}(x - y^2)^2 + 4z^2y^2 &= (x + y^2)^2 \\ 4z^2y^2 &= (x + y^2)^2 - (x - y^2)^2 = (2x)(2y^2) \\ z^2 &= x\end{aligned}$$

So now our triple is $(z^2 - y^2, 2zy, z^2 + y^2)$ and we have discovered a general form for Pythagorean triples.

3.1 Challenge questions

1. This does not look like our formula for where the leg and hypotenuse differ by 1. Prove that they give the same values.
2. A primitive Pythagorean triple is one where the $\gcd(a, b, c) = 1$. Find what constraints on y and z we need to have our general Pythagorean triple output primitive Pythagorean triples.
3. Derive a similar formula for quadruples (a,b,c,d) where $a^2 + b^2 + c^2 = d^2$
4. Generalize your results from above even further by finding a n -tuple such that $a_0^2 + a_1^2 + \dots + a_{n-1}^2 = a_n^2$
5. An integer which is the area of a right angled triangle with integer sides is called a Pythagorean. Prove that for every positive integer $n > 12$ there exists a Pythagorean number between n and $2n$. (Korea Math Olympiad 1993)