Polygon Angle-Sum Theorem

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1 Introduction

1.1 What is the Polygon Angle-Sum Theorem?

The polygon angle-sum theorem is a theorem that finds the sum of all the angles in an n-sided polygon. The basic form of the theorem is:

$S = 180(n-2)$, where S is the sum and n is the number of sides in the polygon.

To find the angle measure, we simply divide both sides by n in the previous equation (Note that this only applies for regular polygons). We can also manipulate the previous equation to find that the sum of the exterior angles, which is given by the formula $\frac{360}{n}$.

Surprisingly enough, this basic theorem is very useful when it comes to angle chasing problems in every competition, but mostly in MATHCOUNTS competitions.

1.2 Proving the Polygon Angle-Sum Theorem

(Note: I have not provided a diagram because of the ambiguity of this theorem. Try to prove it a different way!)

Imagine that you have a random polygon of any side length. Now draw in all of the diagonals from a single point. You can only draw $(n-2)$ diagonals (where n is the number of sides), because you can't draw diagonals to the two adjacent points. This means that you will have also formed $(n-2)$ triangles. Notice that since all of the angles of the triangles compose all of the angles of the polygon, we can definitely say that the sum of all of the angles in all of the triangles equals the sum of all of the angles in the polygon. Therefore, we have now proved that the sum of the angles in an n-sided polygon is $\boxed{180(n-2)}$.

2 Using the Polygon Angle-Sum Theorem

As I said before, the main application of the polygon angle-sum theorem is for angle chasing problems.

2.1 MATHCOUNTS 2015 Chapter Sprint Problem 30

For positive integers n and m, each exterior angle of a regular n-sided polygon is 45 degrees larger than each exterior angle of a regular m-sided polygon. One example is $n = 4$ and $m = 8$ because the measures of each exterior angle of a square and a regular octagon are 90 degrees and 45 degrees, respectively. What is the greatest of all possible values of m?

Using the exterior angle formula that you proved earlier, we can form the equation

$$\frac{360}{n} = \frac{360}{m} = 45$$

We can subtract $\frac{360}{m}$ from both sides to get $\frac{360}{n} - \frac{360}{m} = 45$. From here, we can divide both sides by 360 to get $\frac{1}{n} - \frac{1}{m} = \frac{1}{8}$. By quickly using guess and check, we can see that there are very few positive values that satisfy the equation. Therefore, the largest value of m is $\boxed{56}$.

2.2 MATHCOUNTS Mini 17 Problem 2

The measures of the interior angles of a convex hexagon form an increasing arithmetic sequence. How many such sequences are possible if the hexagon is not equiangular and all of the angle degree measures are positive integers less than 150 degrees?

By plugging in 6 into the angle-sum theorem, we know that all of the angles in a hexagon add up to 720 degrees. We can set up an equation using this information. Assuming that d is the common difference between each term, we can say that

$$6x + 10d = 720, \text{ where } x \text{ is the smallest angle measure.}$$

From here, we can divide both sides by 2 to get that $3x + 5d = 360$. Looking at the equation, we can see that since 5 is not divisible by three, d has to be a multiple of 3 for all of the terms to be integers. We also know that the angles cannot be greater than 150 and that they are not equiangular. Therefore, we have a total of $\frac{147-120}{3} = \boxed{9}$ sequences.

3 Closing Thoughts

I hope this article was helpful in learning about the polygon-angle sum theorem! As I said before, this theorem is applicable in many types of problems.