Logarithms

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1 Introduction

Logarithms are a shortcut for exponents, which are a shortcut for multiplication (so technically logarithms are a shortcut for multiplication). Logarithms are expressed as $\log_a b$; a is the **base** of the logarithm and b is the **argument** of the logarithm. **The base and argument of a logarithm have to be positive**. There are two special cases of logarithms; when $a = 10$, we write the logarithm as $\log b$ and when $a = e$, we write the logarithm as $\ln b$ (for more information on the number e, visit the link at the end of the article). Say we have the equation $a^x = b$. We can rewrite this as $x = \log_a b$. What this means is when a is raised to the xth power, the result is b. In this article, we will learn how to solve basic logarithmic equations, prove many properties of logarithms, and apply our knowledge to some problems.

2 Solving Basic Logarithmic Equations

We'll start our study of logarithms by solving a basic logarithmic equation to get a feel for logarithms.

Solve the equation $\log_2 32 = x$.

We can rewrite the equation as $2^x = 32$. Since $2^5 = 32$, $\boxed{x = 5}$.

2.1 Exercises

1. For what value of x is the equation $\log_3 x = \log_3(3x - 8)$ true?

2. What is x^2 if $x = \log_2 4$?

3 Logarithmic Properties

In this section, we will prove the many logarithmic properties. One of the main things you should learn from the following proofs is the extreme importance of the simple tool of setting $\log_a b = x$.

3.1 The Addition Property

Prove that $\log_a b + \log_a c = \log_a bc$.

Let's tackle the left-hand side term-by-term. Let $\log_a b = x \Longrightarrow a^x = b$ and let $\log_a c = y \Longrightarrow a^y = c$. Multiplying these two equations gives $a^{x+y} = bc$.

Taking $\log_a$ to both sides, we get $x + y = \log_a bc$. However, since $x = \log_a b$ and $y = \log_a c$, we have proved that $\log_a b + \log_a c = \log_a bc$.

3.2 The Subtraction Property

Prove that $\log_a b - \log_a c = \log_a \frac{b}{c}$.

We proved the last property by setting $\log_a b$ and $\log_a c$ each to a different variable. We try to use this strategy again. Let $\log_a b = x \implies a^x = b$ and $\log_a c = y \implies a^y = c$. Dividing the first equation by the second equation, we get $\frac{a^x}{a^y} = \frac{b}{c} \implies a^{x-y} = \frac{b}{c}$. Taking $\log_a$ to both sides, we get $x - y = \log_a \frac{b}{c}$. However, since $x = \log_a b$ and $y = \log_a c$, we have proved that $\log_a b - \log_a c = \log_a \frac{b}{c}$.

3.3 The Argument-Power Property

Prove that $\log_a b^n = n \log_a b$.

Once again, we let $\log_a b = x \implies a^x = b$. Raising both sides to the nth power, we have that $a^{xn} = b^n$. Taking $\log_a$ to both sides, we get $xn = \log_a b^n$. However, since $x = \log_a b$, we have proved that $n \log_a b = \log_a b^n$.

3.4 The Inverse Property

Prove that $\log_a b = \frac{1}{\log_b a}$.

Again, we let $\log_a b = x \implies a^x = b$. Taking $\log_b$ to both sides, we get $\log_b a^x = 1 \implies x \log_b a = 1 \implies x = \frac{1}{\log_b a}$. However, since $x = \log_a b$, we have proved that $\log_a b = \frac{1}{\log_b a}$.

3.5 The Power Property

Prove that $\log_{a^n} b^n = \log_a b$.

This time, we let $\log_{a^n} b^n = x \implies a^{xn} = b^n$. Taking the nth root of both sides we get $a^x = b$. We can rewrite this as $\log_a b$, so we have proved that $\log_{a^n} b^n = \log_a b$.

3.6 The Change-of-Base Formula

Prove that $\frac{\log_c b}{\log_c a} = \log_a b$.

We let $\log_a b = x \implies a^x = b$. Taking $\log_c$ to both sides, we have that $x \log_c a = \log_c b \implies x = \frac{\log_c b}{\log_c a}$. However, since $x = \log_a b$, we have proved that $\log_a b = \frac{\log_c b}{\log_c a}$.

3.7 Exercises

1. Prove that $(\log_a b)(\log_b c) = \log_a c$ (this is called the **chain rule**).
2. Prove that $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$.
3. If I only have a calculator that evaluates base-10 logarithms, is it possible for me to evaluate $\log_2 3$? If it is possible, how so?

4 Applications

4.1 Simple Equations

- Find all x such that $\log_2(x+2) + \log_2 x = 3$.

We can simplify the equation to $\log_2 x(x+2) = 3 \implies x(x+2) = 8$. Solving the quadratic, we get the two solutions for x as -4 and 2. However, since x has to be positive, -4 is extraneous, so our only solution is $\boxed{x = 2}$.

- Simplify $\log_4 3 + \log_4 6 - \log_4 9$.

We can rewrite the equation as $\log_4\left(\frac{3\times 6}{9}\right) \implies \log_4 2 = \boxed{\dfrac{1}{2}}$.

4.2 Exercises

1. Find all x such that $\log 6x - \log(x+3) = \log(x+1)$.
2. What is the value of $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)\cdots(\log_{62} 63)(\log_{63} 64)$?

4.3 Advanced Equations

- Find all the solutions to the equation $x^{\log x} = \frac{x^3}{100}$ (source: AHSME 1962).

Taking $\log_x$ to both sides results in $\log x = \log_x \frac{x^3}{100}$. We can rewrite the right-hand side as $\log_x x^3 - \log_x 100$, which simplifies further to $3 - 2\log_x 10$. Our equation becomes $\log x = 3 - 2\log_x 10$. Remembering the Inverse Property, we rewrite $\log_x 10$ as $\frac{1}{\log x}$. We now have $\log x = 3 - \frac{2}{\log x}$. We can set $y = \log x$, so we have $y = 3 - \frac{2}{y} \implies y^2 - 3y + 2 = 0$, so $y = 2$ and $y = 1$. These two values of y result in x-values of 100 and 10, respectively. Our answers are $\boxed{10}$ and $\boxed{100}$.

- For all positive numbers $x \neq 1$, simplify $\frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \frac{1}{\log_5 x}$ (source: AHSME 1978).

The Inverse Property helps us rewrite the equation as $\log_x 3 + \log_x 4 + \log_x 5$. We can further simplify this to $\log_x(3\cdot 4\cdot 5) = \boxed{\log_x 60}$.

4.4 Exercises

1. Suppose that p and q are positive numbers for which $\log_9 p = \log_{12} q = \log_{16}(p+q)$. What is the value of $\frac{q}{p}$ (source: AHSME 1988)?
2. If $60^a = 3$ and $60^b = 5$, then find $12^{[(1-a-b)/(2(1-b)]}$ (source: AHSME 1983).
3. If $\log 36 = a$ and $\log 125 = b$, express $\log \frac{1}{12}$ in terms of a and b (source: MAθ 1992).

5 Closing Thoughts

I hope this guide to logarithms and their applications was helpful! If you want to learn more about logarithms, check out some of the following links.

6 Links for Additional Information

- http://aops-cdn.artofproblemsolving.com/products/aops-vol2/exc1.pdf
- http://artofproblemsolving.com/wiki/index.php/E
- http://artofproblemsolving.com/wiki/index.php/Logarithm