Law Of Cosines

uwotm8

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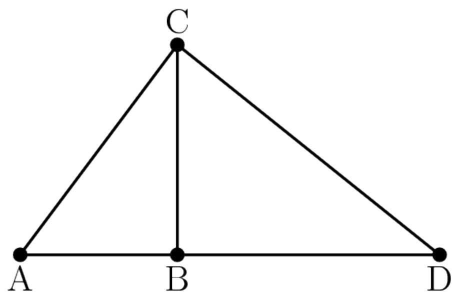
1 Introduction

1.1 What is the Law of Cosines?

The law of cosines, more commonly referred to as "LoC", is a very powerful tool that allows you to solve for missing sides and angles, given the SAS or SSS congruence. It is a common problem solving strategy for later problems on the AIME, and is really helpful to know. In it's simplest form, the law of cosines is:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

1.2 Proving the Law of Cosines



We want to start by equating the lengths of the sides of △ABC and △BCD. By using simple trigonometry, we can start out by saying that

$$CB = AC \sin A$$

and that

$$AB = AC \cos A$$

Using the pythagorean theorem on △BCD, we have that $BD^2 + BC^2 = CD^2$. Since BD is simply $AD - AB$, we can substitute in our previous equations to get

$$CD^2 = AC^2 \cdot \sin^2 A + (AD - AC \cdot \cos A)^2$$

We can now expand the terms in the parentheses to get

$$CD^2 = AC^2 \cdot \sin^2 A + AD^2 + AC^2 \cdot \cos^2 A - 2ab \cdot \cos A$$

We can now factor together the two terms with the AC^2 to get

$$CD^2 = AC^2(\sin^2 A + \cos^2 A) + BC^2 \text{ - } 2ab \cos A$$

Since $\sin^2 A + \cos^2 A = 1$, our expression finally reduces to

$$CD^2 = AC^2 + BC^2 - 2ab \cdot \cos A$$

Although this may not look like the equation I presented above, this is the same equation. Think of it in this way:

- AB and AD are a and b respectively, where they are the sides next to $\angle C$.
- CD is c, which is the side opposite of $\angle C$.
- $\angle C$ is the angle opposite side c and between sides a and b

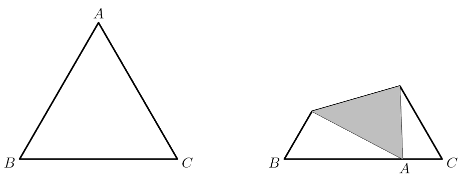
And now, we are done proving the law of cosines.

2 Using the Law of Cosines

As I said before, the law of cosines is very useful and helpful when solving AIME geometry problems usually from problems 9-15.

2.1 2013 AIME I Problem 9

A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on $\overline{BC}$ a distance 9 from point B. The length of the line along which the triangle was folded can be written as $\frac{m\sqrt{p}}{n}$, where m, n and p are all integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find the value of $m + n + p$.



From the picture, we can conclude that the triangle shaded in grey is congruent to the triangle that is cut off from the folded picture. We can call the point of the grey triangle that intersects $\overline{AB}$ point D and we can call the point that intersects $\overline{AC}$ point E. We will also refer to A' as the point on $\overline{BC}$ Since $A'D = AD = 12 - BD$, we can use law of cosines to find the missing sides of △A'BD. This is especially useful since ABC is equilateral and all angles are 60°. We can now form the equation

$$(12 - BD)^2 = BD^2 + 81 - 18BD \cos 60°$$

We can expand the left hand side to get $144 + BD^2 - 24BD$. From here, we can cancel out the BD^2 terms and isolate the variable. Since cos $60° = \frac{1}{2}$, our new equation looks like

$$63 = 15BD$$

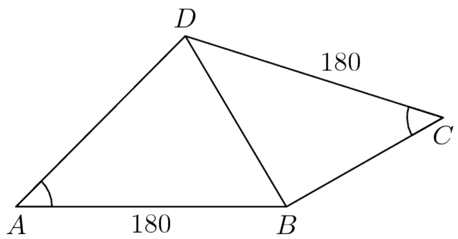
Dividing both sides by 15 gives us that $BD = \frac{21}{5}$. This means that $A'D = \frac{39}{5}$. We can perform the same steps on △A'CE to get that $EC = \frac{45}{7}$ and that $A'E = \frac{39}{7}$. Since △ADE ≅ △A'DE, therefore ∠A = ∠C = 60° We can apply the law of cosines on △A'DE to get

$$DE^2 = \left(\frac{39}{5}\right)^2 + \left(\frac{39}{7}\right)^2 - \left(\frac{39}{5}\right) \cdot \left(\frac{39}{7}\right)$$

Solving for DE gives us that $DE = \frac{39\sqrt{39}}{35}$. Since the question is asking for $m + n + p$, our final answer is $39 + 39 + 35 = \boxed{113}$. And we are done.

2.2 2003 AIME I Problem 12

In convex quadrilateral $ABCD$, $\angle A \cong \angle C, AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\lfloor 1000 \cos A \rfloor$. (The notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.)



Looking at the quadrilateral, if we draw in $\overline{BD}$, we can create two equations that relate the sides of triangles ABD and BCD. Our equations are

$$BD^2 = 180^2 + AD^2 - 360(AD) \cdot \cos A$$

and

$$BD^2 = 180^2 + BC^2 - 360(BC) \cdot \cos C$$

Since $\angle A \cong \angle C$, we can replace the angle in either equation so that it matches the latter. We can set both equations equal to get

$$180^2 + AD^2 - 360(AD) \cdot \cos A = 180^2 + BC^2 - 360(BC) \cdot \cos A$$

From here, we can first cancel out the 180^2 terms,as they appear on both sides. Since we now want to isolate the cosine terms, we put them on one side and the square terms on the other side to get

$$AD^2 - BC^2 = 360(AD - BC) \cos A$$

We can factor out $(AD - DC)$ from the left hand side and divide to get

$$AD + BC = 360 \cos A$$

In AIME problems, we want to use all of the information given in the question. Looking back at the question, we have not used the perimeter of the quadrilateral to find anything. We can set up the equation $AD + BC + AB + CD = 640$. Since $AB = CD = 180$, we can plug that back in to get that $AD + BC = 280$. We can substitute that back into the previous equation to get

$$280 = 360 \text{ cos } A$$

Dividing both sides by 360 gives us that

$$\text{cos } A = \frac{7}{9}$$

The fraction $\frac{7}{9} = .\overline{7}$. Therefore, $\lfloor 1000 \cos A \rfloor = \boxed{777}$. And we are done.

3 Closing Thoughts

I hope this paper on the Law of Cosines was helpful! If you wish to learn more, here are some great pages to look at:
- http://www.artofproblemsolving.com/wiki/index.php/Law_of_Cosines
- http://www.khanacademy.org/math/trigonometry/less-basic-trigonometry/law-sines-cosines/v/law-of-cosines-example