CALCULUS

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Chapter 1

What is Calculus?

What is it?

I'm sure you've all already heard about what calculus is from the other students in my calculus class, but just in case you need a refresher, here you go. In my opinion, to create a concise, single sentence definition of calculus that thoroughly conveys the true idea of what calculus is about would be close to impossible. I think instead, that to properly teach the idea behind the essence of calculus (what calculus really is) it would require much more than just a sentence. But seeing as I did not choose to present to you all solely about what calculus is, I'll give it my best shot to sum it up in one sentence. As simple as I can make it, Calculus is the branch of mathematics that deals with rate of change and accumulation. If you don't believe that Calculus is important or useful in the slightest, hopefully I can change your mind a little bit before I begin the actual lesson.

Why is it Useful?

Rather than give you the exact details, I'll simply tell you where Calculus is used in the real world. For starters, math related occupations use calculus such as actuaries, mathematicians, and statisticians. Calculus is also used in nearly every single type of engineering you can think of, Biological Science, Medical Science, Physical Science, Economists, Social Scientists, Medical and Health Service Managers, Roofers, and even Funeral Directors. One of my personal favorite things about Calculus is that it's the first time you'll realize you don't need anywhere near the amount of information normally given in problems to solve them!

To Infinity and Beyond!

One of the recurring themes in calculus is the concept of infinity. Students usually have a hard time (at least in my opinion :P) wrapping their head around it, so I'm going to discuss that as part of this presentation. Infinity isn't actually a number, it is, as I previously mentioned, a concept. There's obviously no way to have an infinite amount of something. In addition, dealing with infinity as a number can lead to some wacky false results in math. Lets see what would happen if we treated infinity as a number. If we were to keep adding the counting numbers we would get infinity. So

$$1 + 2 + 3 + 4 + ... = \infty$$

Now we're assuming ∞ is a number, so let's subtract 1 from both sides. This results in

$$2 + 3 + 4 + ... = \infty - 1.$$

But we also know that

$$2 + 3 + 4 + ...$$

is infinite! So we have

$$\infty = \infty - 1.$$

Subtracting ∞ from both sides, we get $0 = -1$, which is obviously not true! However mathematicians, being the clever people they are, have found ways to tame infinity. For example, a geometric sequence is a list of numbers in which the previous terms is multiplied by a fixed number to get the next term. Let's consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$$

What would happen if we were to add all of these terms up? In this case, we actually get a number! Each terms is getting smaller and smaller, so we might assume that it can only add so much before it reaches some sort of limit. So let's say that the value of the sum is X. Then we have

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = X.$$

If we multiply both sides by 2, we get

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2X.$$

But we know the value of a huge chunk of this! Let's take a closer look at the second equation I wrote out.

$$2 + \boxed{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...} = 2X.$$

From our first equation we know the boxed part is just X! So substituting we get $2 + X = 2X$. So what is X? You should be able to answer that for yourself! :) In this sense, we can actually use the concept of infinity to produce extremely accurate results. Say you wanted to pay a cashier the exact amount that you were charged. If you only had a 100 dollar bill, your goal probably wouldn't be accomplished. But if instead you had ten 10 dollar bills, you would have a higher chance of being able to pay him. Even better say you had one hundred 1 dollar bills. What would be even better than that? If you had ten thousand pennies, then you would be able to pay the cashier exactly what you were charged. Basically, one of the big ideas of calculus is looking at extremely large amounts of things to better the approximation dealing with that thing. The more and more you have, the closer your approximation gets. Or the less you have of something, the closer it gets.

Limits

In my previous example with the geometric series, the sum gets closer and closer to the number 2, but it doesn't actually ever reach that number. Technically speaking, the sum will always be a *very very* small amount away from the number 2. But what we can do, is we can say that as the amount of terms approaches infinity, the sum gets infinitesimally close to the number 2. Mathematically, this is called the limit. If we want to write this formally, we would say

$$\lim_{\#\text{of terms}\to\infty} 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 2.$$

Limits are essentially how we turn approximations into exact values.

So What's the Area of a Circle?

We all know the area of a circle is

$$\pi r^2$$

Pretty easy huh? But does anyone know why? How do you actually come up with that formula that seems so well known? Well let's try to figure it out together. In math we always try to break down complex things into simpler things that we know how to deal with. One of my favorite quotes:

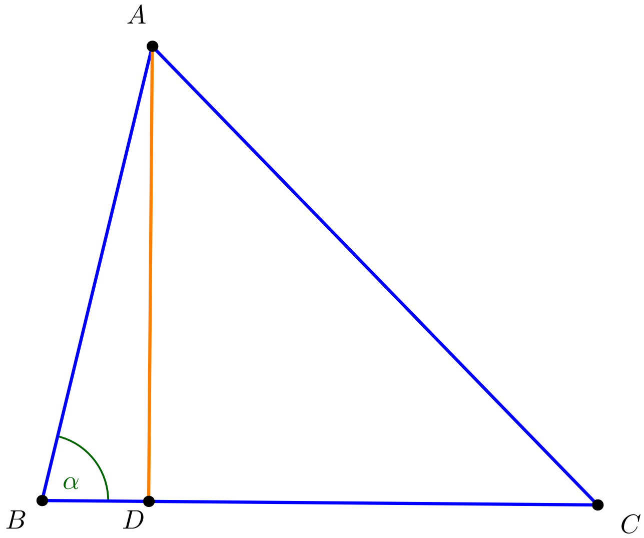
> "The essence of mathematics is not to make simple things complicated, but to make complicated things simple." S. Gudder

Breaking it Down

How do you find the area of a triangle? Most probably everyone only remembers the classic

$$\frac{\text{base} \cdot \text{height}}{2}$$

. However, we can also express the area slightly differently, which I'm going to require for deriving the area of a circle.



In the diagram above, we have $\triangle ABC$ with $\angle ABC = \alpha$ and height AD. Now using the simple area of a triangle, we can say that the area of $\triangle ABC$ is

$$\frac{BC \cdot AD}{2}$$

However, if we can recall some of our trigonometry knowledge, we have

$$\sin\alpha = \frac{AD}{AB}$$

or

$$AB\sin\alpha = AD$$

Plugging this in we get that the area of the triangle is simply

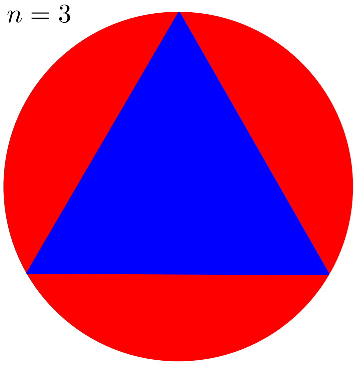
$$\frac{BC \cdot AB\sin\alpha}{2}$$

.

Getting Back To The Circle

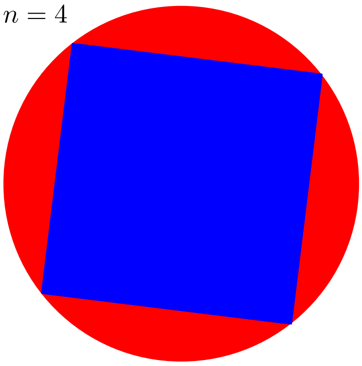
To approximate the area of a circle, I'm going to use regular polygons. Regular polygons are just shapes that have equal sides and equal angles. As I previously mentioned, the more and more of something we have, the more accurate it gets (usually). Let's look at the diagrams below.

$n = 3$

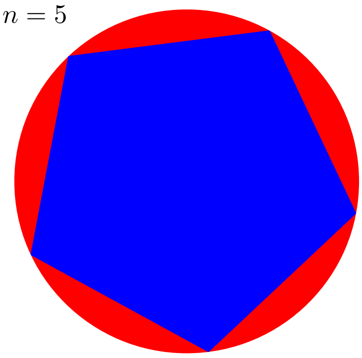


In this diagram, a regular triangle is inscribed within the circle (a regular triangle is just an equilateral triangle). As you can see, the triangle is a pretty terrible approximation of the area. The area of the triangle is in blue, and the remaining area is in red. In the following diagrams, the sides of the polygons will gradually increase. Be sure to note what happens to the red area (the difference between the area of the shape and the circle).

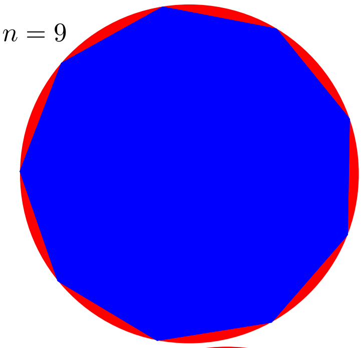
$n = 4$



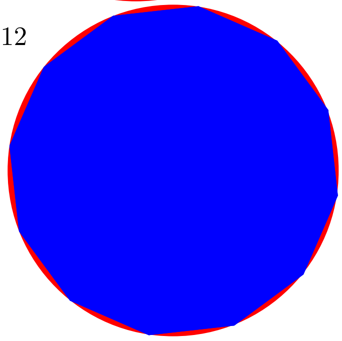
$n = 5$



$n = 9$

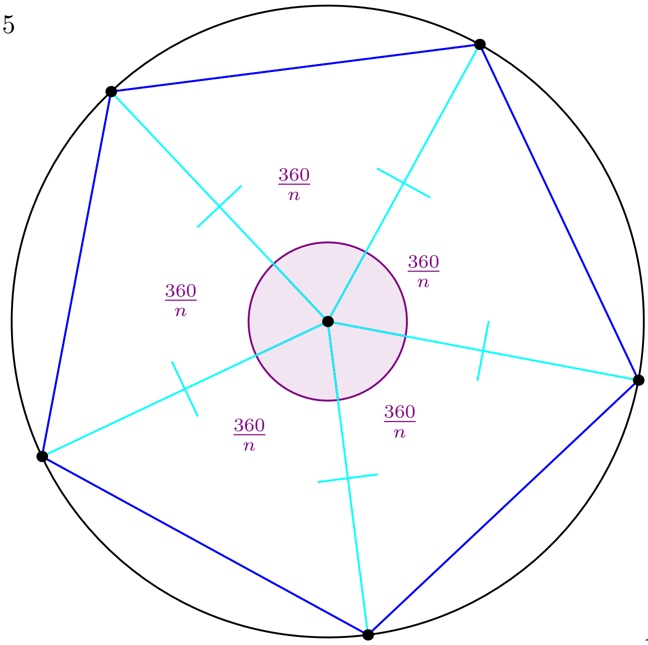


$n = 12$



Aha! As we increase the amount of sides of the polygon, the closer and closer it gets to the area of the circle. But there's still a problem. We don't know how to deal with increasing the sides of the polygon and it's relation with area :(So instead, lets try to make it into something we do know. We can split up a polygon into smaller triangles with one vertex at the center of the circle and the others at the vertices of the polygon.

$n = 5$



As we figured out before, for us to make the shape as close to possible as a circle, we want it to have as many sides as possible. So we want to take the number of sides n, and make it approach infinity.

There are n triangles and the central angle formed by each triangle will be $\frac{360}{n}$ because there are n sides, and each of the blue segments has length r, where r is the radius of the circle. So if we're going to use our triangle method, making n approach infinity would make the central angle approach 0. The reason for this is because the larger and larger n gets, the smaller and smaller $\frac{360}{n}$ gets (dividing by a larger number results in a smaller fraction). Now keeping in mind all these facts, we can go ahead and try to express the area of this polygon using individual triangles, and at the same try to maximize that area. The area of the polygon will be equal to

$$\frac{1}{2}r^2 \sin \frac{360}{n} \cdot n$$

Now here is where it gets a bit complicated. We want to take the limit of this expression as $n \to \infty$ in order to make it as close to a circle as possible (increasing the # of sides , remember?). But **evaluating** limits is something you'll learn in calculus. Anyways, we proceed.

$$\lim_{n\to\infty} \frac{1}{2}r^2 \sin \frac{360}{n} \cdot n$$

The above expression will essentially evaluate the area of a polygon with infinite sides, or a circle. But to evaluate we must use a clever trick. We let

$$\frac{1}{n} = x$$

. Then as

$$n \to \infty$$

$$x \to 0$$

So the expression becomes

$$\lim_{x\to 0} \frac{r^2 \sin 360x}{2x} = 180 \lim_{x\to 0} \frac{r^2 \sin 360x}{360x}$$

Now we can let $360x = y$, and it further simplifies to

$$\lim_{y\to 0} \frac{r^2 \sin y}{y}$$

One of the special limits you will eventually learn in Calculus goes as follows:

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Using this we have the area of the circle is equal to

$$180 \lim_{y\to 0} \frac{r^2 \sin y}{y} = 180r^2$$

. But that doesn't look quite right. 180 is in degrees, so we must convert this to radians to get our final answer. If you remember, we have the following relation

$$180 = \pi$$

Thus the area of the circle is

$$\pi r^2$$

There you have it! We successfully derived the area of a circle! The power of Calculus... :)

Another thing that Calculus is useful for is finding instantaneous rates. If you remember, in precalculus you are taught how to graphically represent the average rate of change by drawing a secant line through two points on the graph, then finding the slope of that line. The instantaneous rate of change however, is the slope of the tangent line at the point you are finding the rate. Now again, the same concept appears! If you were to take the second point on the secant line and keep on moving it closer and closer to your first point, eventually you would get the tangent line.

The Derivative

Let's start of this section with a little notation. Δx is defined as the change in x and $f(x) = y$. Now let's say we selected two points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$. The slope of the line between these two points would be

$$\frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Now because we want to make this the instantaneous rate of change, we want the two points to be as close together as possible. In other words, we want to make the change between the x values, Δx as small as possible. So we set up the following expression for the instantaneous rate of change

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

This expression is known as the derivative. The most common notations used for the derivative are

$$\frac{dy}{dx}$$

$$f'(x)$$

and

$$y'(x).$$

Fortunately enough, people have come up with ways to simplify this extremely tedious/tiresome computation of the derivative. Instead of using the limit definition, there now exist shortcuts (although you should never just memorize, always know why they are what they are). For example, if I were to ask you to evaluate the derivative of x^4, then there would be two ways to do it. One would include using the limit definition of a derivative, and the other would be using the shortcut. The shortcut says if you have some function of the form

$$f(x) = ax^n$$

then

$$\frac{dy}{dx}f(x) = anx^{n-1}$$

.

Anyways, I'm not going to spend too much time going in depth on these things, because you'll learn them once you get to Calculus.

Chapter 2

Introduction to Integrals

That Looks Like a Stretched Out S

The integral is basically the opposite of the derivative. Just as division is to multiplication, the integral is to the derivative. The integral is an accumulation function, and is denoted by what looks like a stretched out S. The symbol for the integral is

$$\int f(x)dx$$

It must always be enclosed as such.

Using Integrals

One of the most basic functions of integrals is finding area underneath curves. Again, to build an idea of what integrals do we must first start off with our favorite concept! Let's take a look at a random curve $f(x) = x^2$, and say we want to find the area underneath this curve from when $x = 0$ to when $x = 8$. Now we need someway to approximate this area, because of course we don't know of any other way to deal with it! Generally in math we try to make things as simple as possible, so to approximate the area, we just choose rectangles. When there are 2 rectangles, each with width 4, we see that our guess for the area is pretty off (just as we saw with the circle). So what should we do to make our approximation more accurate? We increase the number of rectangles. So let's take a look at 4 rectangles, each with width 2. Better! But still, its missing a lot of area. I'm pretty sure you get the idea now. Here's a summary of a few of the basic integrations formulas you'll learn.

- $\int kf(u)du = k\int f(u)du$
- $\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$
- $\int du = u + C$
- $\int u^n du = \frac{u^{n+1}}{n+1} + C$

On another note, when integrals have bounds (numbers below them and above them) that tells you from where to where you're finding the area, and they are called **definite integrals**. When they do not have bounds, they are called **indefinite integrals**. To evaluate an integral with bounds, you first evaluate it as it would be without bounds, then plug in each of your bounds and subtract the lower one from the upper one. For example: Evaluate

$$\int_0^3 x^2 dx$$

The solution to this would be to first evaluate

$$\int x^2 dx$$

which we find to be (using the fourth item)

$$\frac{x^3}{3}.$$

Then we have

$$\left[\frac{x^3}{3}\right]_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = \boxed{9}$$

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Where Algebra 2 and Precalc Come Into Play

Do you remember your trig. formulas or partial fraction decomposition? Well you're going to have to for AP Calc. Here's an example of one of the questions you could encounter dealing with trigonometric identities. Evaluate

$$\int_0^\pi \cos^2(\theta) d\theta$$

For this question you must know that

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

Another possible example of an integration question would be: Find the indefinite integral

$$\int \frac{1}{4x^2 - 1} dx$$

For this one it is necessary to know partial fraction decomposition. This would be the first step:

$$\frac{1}{4x^2 - 1} = \frac{A}{2x - 1} + \frac{B}{2x + 1}$$

From here we would rearrange it to get the equation

$$A(2x + 1) + B(2x - 1) = 1$$

and then we would solve for A and B.

So that's an introduction to calculus, things you'll encounter, and the uses of the subject. I hope you enjoyed it!