The Area of a Triangle

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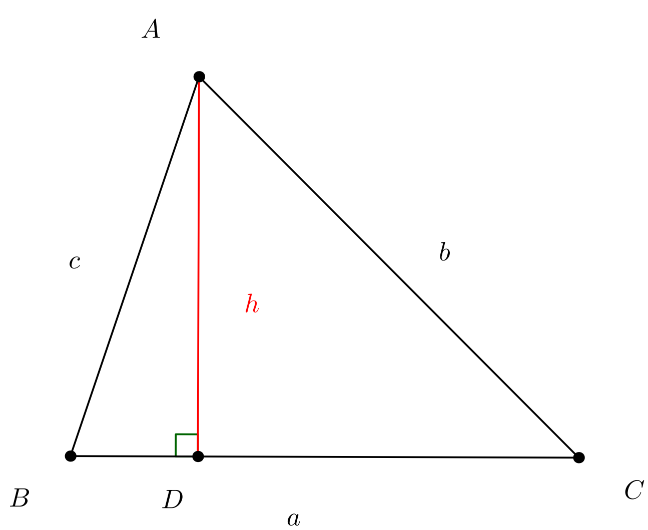
1 Introduction

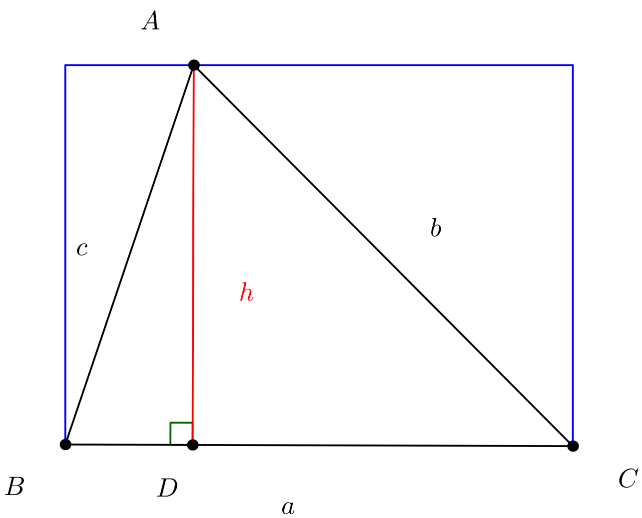
In this article we will be discussing the various methods used for determining the area of a triangle. Let $[X]$ denote the area of $\triangle X$.

2 Using Base and Height

To start off, the simplest method for finding the area of a triangle is to multiply the base of the triangle by the height to that base, and then divide the result by 2. For example, consider the following triangle:



The area of $\triangle ABC$ would then be $\dfrac{BC \times AD}{2} = \dfrac{a \times h}{2}$. How do we prove this? Well the area of this triangle is simply half the area of a rectangle with the same sidelength and height. Let's take a look at the diagram below:

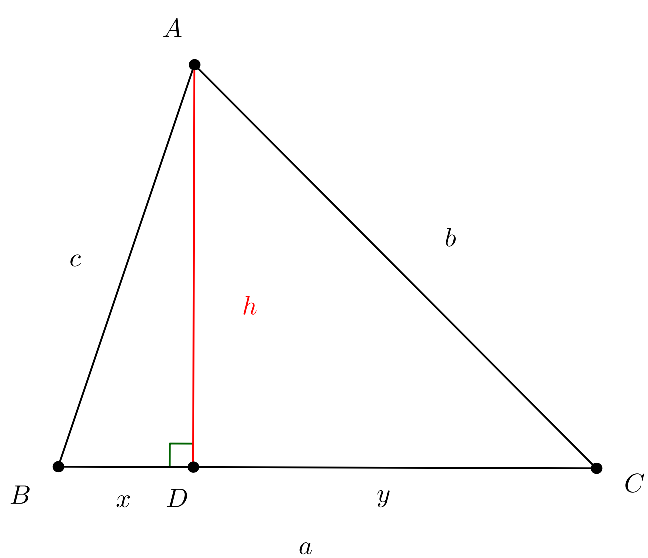


As you can see, we have two copies of each of the right triangles, and together they make up the rectangle. So only one copy of each would produce half the area of the rectangle. We know the area of a rectangle is $l \times w$, so the area of the triangle would be $\dfrac{l \times w}{2}$. But l is the base and w is the height. Thus we have the area is $\dfrac{\text{base} \times \text{height}}{2}$.

However, we're not always given the height of the triangle, and that can be problematic if we are still asked to find the area.

2.1 Determining the Height

Let's say we were only given the lengths of the sides of the triangle. As of this moment, we don't know how to find the area. But we do know that the only way we *can* find the area is by first finding the height. For this, we're going to have to call our old friend, the Pythagorean Theorem.



Consider the following diagram:

We let the altitude AD split BC into two segments of length x and y as shown. Using the pythagorean theorem and the simple segment addition postulate, we have

$$\textbf{(1)}\ \ x^2 + h^2 = c^2$$

$$\textbf{(2)}\ \ y^2 + h^2 = b^2$$

$$\textbf{(3)}\ \ x + y = a$$

Subtracting **(1)** from **(2)** and using difference of two squares we get

$$(y-x)(y+x) = b^2 - c^2$$

Plugging in **(3)** we have

$$(y-x)a = b^2 - c^2$$

$$y - x = \frac{b^2 - c^2}{a}$$

Adding this to **(3)** results in

$$2y = \frac{b^2 - c^2 + a^2}{a}$$

$$y = \frac{b^2 - c^2 + a^2}{2a}$$

Using **(2)** we get

$$h^2 = b^2 - y^2$$

$$h^2 = (b-y)(b+y)$$

$$h^2 = \left(\frac{2ab - b^2 + c^2 - a^2}{2a}\right)\left(\frac{2ab + b^2 - c^2 + a^2}{2a}\right)$$

$$h^2 = \left(\frac{c^2 - (a-b)^2}{2a}\right)\left(\frac{(a+b)^2 - c^2}{2a}\right)$$

$$h^2 = \frac{(c-a+b)(c+a-b)(a+b-c)(a+b+c)}{4a^2}$$

Thus, taking the square root we have

$$\boxed{h = \frac{\sqrt{(c-a+b)(c+a-b)(a+b-c)(a+b+c)}}{2a}}$$

3 Heron's Formula

We managed to determine the height of the triangle solely in terms of the side lengths, which means we can find the area solely in terms of the side lengths. But instead of writing out that nasty long expression all the time, we can find a way to make it look much nicer, which will not only save time, but will also look more visually appealing. We define the **semiperimeter** of the triangle to be half of the perimeter, or $\frac{a+b+c}{2}$. Now notice that

$$2s = a + b + c$$

$$2(s-a) = b + c - a$$

$$2(s-b) = a + c - b$$

$$2(s-c) = a + b - c$$

Pluggin these in to the expression for the height we get

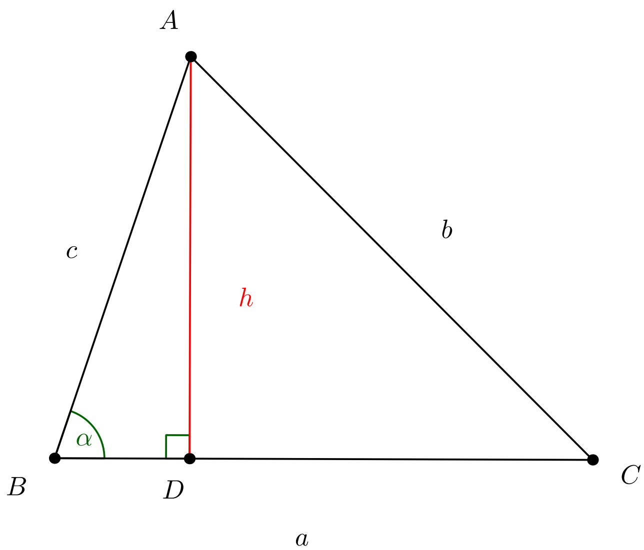
$$h = \frac{\sqrt{16s(s-a)(s-b)(s-c)}}{2a} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$$

Multiplying by the base and dividing by 2 results in

$$\boxed{[\triangle ABC] = \sqrt{s(s-a)(s-b)(s-c)}}$$

4 Area of a Triangle With Sine

Just like Heron's, this method of computing triangle area is also an extremely useful tool. In fact, this combined with Law of Sines was essentially all that was required for an IMO question in 2007!



Let's refer to this triangle again. Let $\angle ABD = \alpha$. Now recall that

$$[\triangle ABC] = \frac{a \times h}{2}$$

However we have that

$$\sin\alpha = \frac{h}{c}$$

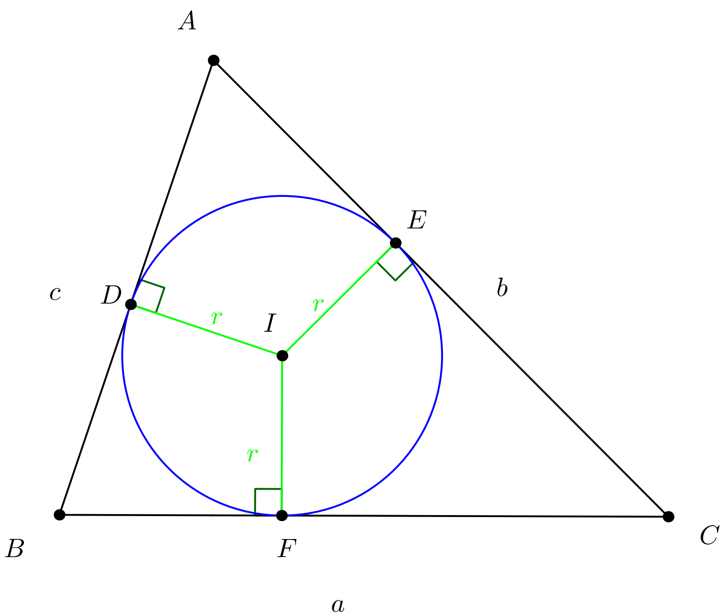
$$c\sin\alpha = h$$

Pluggin this in we get that

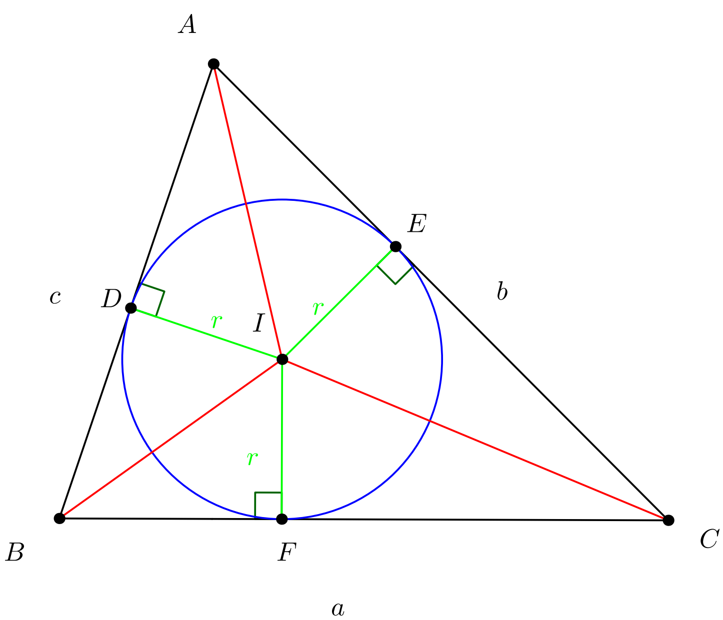
$$\boxed{[\triangle ABC] = \frac{ac\sin\alpha}{2}}$$

5 Area With Inradius

The **incircle** of a triangle is defined as the circle that is tangent to each of the sides of the triangle. Because it is tangent to the sides, if we connect the center of the circle to the tangency points, we will obtain right angles. Here is a diagram:



Now to find the area of $\triangle ABC$ in terms of r, we can simply draw lines from I to points A, B, and C. We will then separately calculate the area of the three triangles formed and add them.



Using the simple area of a triangle (base and height definition) we have the following:

$$[\triangle ABI] = \frac{c \times r}{2}$$

$$[\triangle BIC] = \frac{a \times r}{2}$$

and

$$[\triangle CIA] = \frac{b \times r}{2}$$

Adding all of these together gives us

$$[\triangle ABC] = r\left(\frac{a+b+c}{2}\right)$$

But recall that $\dfrac{a+b+c}{2} = s$, where s is the semiperimeter. So we have

$$\boxed{[\triangle ABC] = \frac{r \times s}{2}}$$

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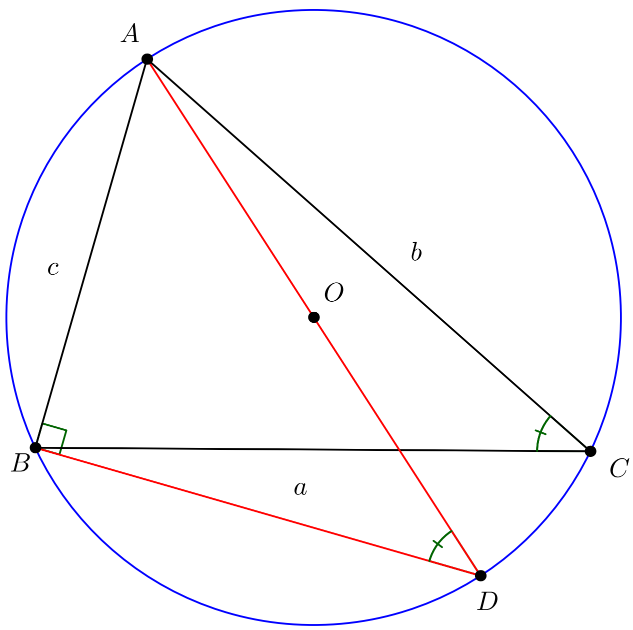
6 Area With Circumradius



To find the area with the circumradius we are going to have to use the **Law of Sines**. As shown above, we start by drawing a line through one vertex and the center of the circle. We then take the intersection point of this line and connect it to another vertex. This forms a $90°$ angle, because the first line is the diameter of the circle. In addition, the angle formed between the diameter and segment is equal to one of the vertex angles. Looking at $\triangle ABD$ we can say that $\sin \angle ADB = \dfrac{c}{2R}$, or $\dfrac{c}{\sin \angle ADB} = 2R$. Plugging in $\angle ADB = \angle ACB$, we get $\dfrac{c}{\sin C} = 2R$. Computing similarly for the other sides results in

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$

.

Now recall that the area of this triangle may be written as $\dfrac{bc \sin A}{2}$. Using the Law of Sines we can manipulate to find $\sin A$. More specifically we have $\sin A = \dfrac{a}{2R}$. Plugging this in will result in

$$\boxed{[ABC] = \frac{abc}{4R}}$$

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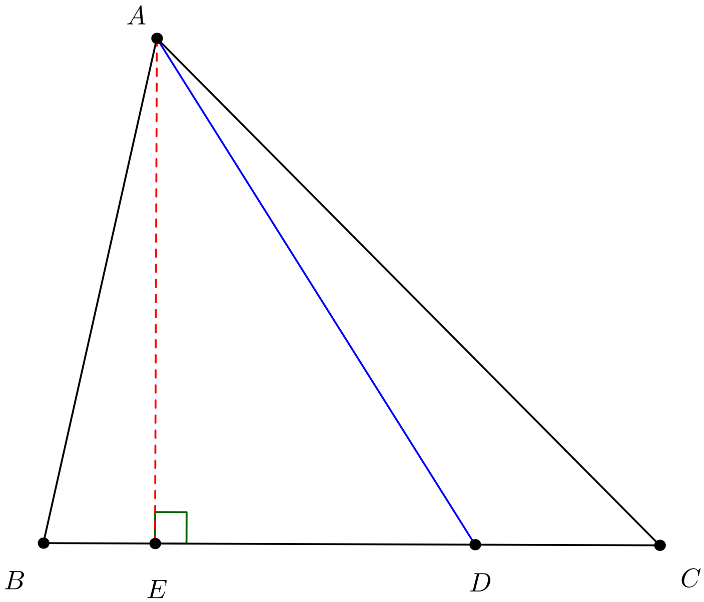
7 Other Useful Techniques

Sometimes we can find the area of a triangle by determining the ratio between that area and a given area of another triangle. For example, let's take a look at this diagram:



Let's say we were given that $[ABD] = 40$ and that $\dfrac{BD}{DC} = \dfrac{5}{3}$, and we were asked to find the area of $\triangle ADC$. Rather than to use one of our earlier learned methods to find the area, we can simply do the following. First note that both triangles have the same height, so the ratio of their areas is in fact just the ratio of their bases! We can now set up a proportion

$$\frac{[ABD]}{[ADC]} = \frac{BD}{DC}$$

$$\frac{40}{[ADC]} = \frac{5}{3}$$

Thus we have

$$[ADC] = 24$$

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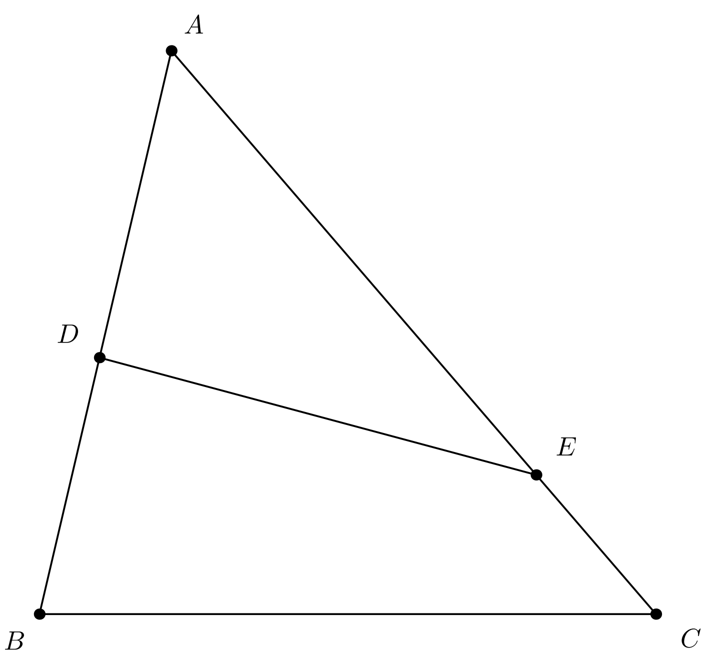
From this idea of ratios we can come up with a nifty yet easy formula. Consider the following diagram:



We have the following equation:

$$\boxed{\frac{[ADE]}{[ABC]} = \frac{AD \times AE}{AB \times AC}}$$

Try to prove it yourself!

8 Problems

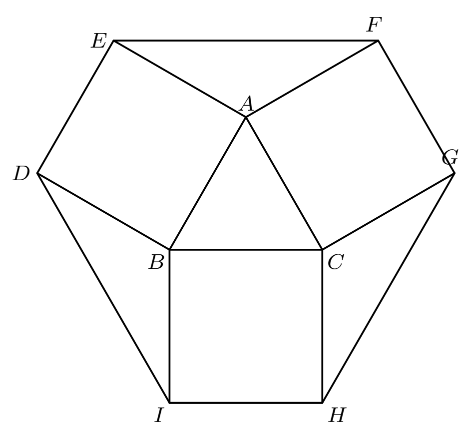
1. In $\triangle ABC$ $AB = 49$ and the height from C to AB has length 20. Find the area of $\triangle ABC$.
2. Let D be the point on side BC of $\triangle ABC$ such that $AD \perp BC$. Let $AB = 17$, $AC = 25$, and $BD = 8$. Compute the area of $\triangle ABC$.
3. In $\triangle ABC$ we have $AB = 13$, $BC = 14$, and $AC = 15$.

   (a) Find the length of the height from A to BC.

   (b) Compute $[ABC]$.
4. In $\triangle ABC$, $AB = 4$, $AC = 2015$, and $\angle BAC = 30°$. Compute $[ABC]$.
5. In $\triangle ABC$, $AB = 13$, $BC = 14$, and $AC = 15$.

   (a) Compute the indradius of $\triangle ABC$.

   (b) Compute the circumradius of the triangle.
6. Points D and E are chosen on segments AB and AC respectively of $\triangle ABC$ 6 units away from point A. Given that $AB = 12$ and $AC = 18$, find the ratio of the area of quadrilateral $BDEC$ to that of $\triangle ABC$.
7. Equilateral $\triangle ABC$ has side length 1, and squares $ABDE$, $BCHI$, $CAFG$ lie outside the triangle. What is the area of hexagon $DEFGHI$?

   (*2014 AMC 10a #13*)
8. Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$? (*2008 AMC 10a #20*)
9. In $\triangle ABC$, side AC and the perpendicular bisector of BC meet in point D, and BD bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of $\triangle ABD$? (*2002 AMC 12a #23*)

10. Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE? (*2009 AMC 12a #20*)

    There's a bunch of area problems, probably at least one on every AMC or AIME you can think of, so if this wasn't enough, feel free to roam AoPS (or anywhere on the internet for that matter) for past comepeitions in order to quench your thirst!