

# An Introduction to Ptolemy's Theorem

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## Contents

<b>1</b>	<b>Introductory questions concerning the handout</b>	<b>2</b>
<b>2</b>	<b>Ptolemy's Theorem - The key of this Handout</b>	<b>2</b>
<b>3</b>	<b>Example Problems</b>	<b>5</b>
3.1	Carnot's Theorem - Noticing Configurations . . . . .	5
3.2	IMO 1995, #5 - Symmetry in Hexagon . . . . .	6
<b>4</b>	<b>Problems - Practice makes perfect</b>	<b>8</b>
<b>5</b>	<b>Hints - Giving up in war</b>	<b>10</b>

## 1 Introductory questions concerning the handout

Ptolemy's Theorem can be powerful in easy problems, as well as in tough Olympiad problems. Often, it is hard to spot the ingenious use of Ptolemy. As there are not many introductions to Ptolemy's Theorem, I dedicated my time to write a fulfilling, but rather easy introduction.

You should read this article, if

- You don't know Ptolemy's Theorem.
- You don't know Ptolemy's Theorem very well.
- You know Ptolemy's Theorem, but you are rusty.
- You are an expert, but still want to learn more. (Or you just want to criticize my failures.)
- You do not know at least 6 proofs of the theorem.
- You want to help me improve my writing skills.
- You just want to make me happy. ♡

And so on! In general, we will find several proofs to Ptolemy's Theorem, discuss a few examples and at last, there will be a bunch of problems to practice for yourself.

The handout itself is not too difficult. It starts from easier problems and goes up to early Olympiad level. Basic geometry understanding suffice to understand this handout. In specific, you should be confident with angle chasing, similar triangles and low tech theorems. Having some knowledge in basic trigonometry wouldn't hurt, though.

## 2 Ptolemy's Theorem - The key of this Handout

### Ptolemy's Theorem

If  $ABCD$  is a (possibly degenerate) cyclic quadrilateral, then

$$|AB| \cdot |CD| + |AD| \cdot |BC| = |AC| \cdot |BD|.$$

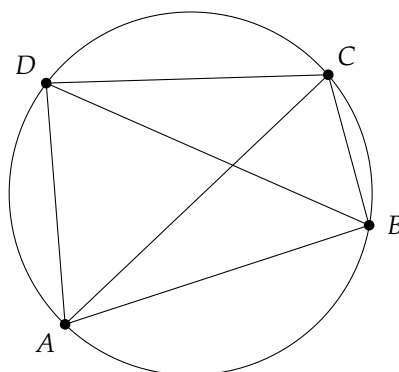


Figure 1: Cyclic quadrilateral  $ABCD$

*Proof.* Spoilers ahead! Try proving it by yourself first, then come back. There are many possibilities to do so.

Pick point  $E$  on line  $BD$ , such that  $\angle DEA = \angle CBA$ . For convenience, call

$$a = |AB|, b = |BC|, c = |CD|, d = |DA|, e = |AC|, f = |BD|.$$

Ptolemy's Theorem would then be rewritten into

$$ac + bd = ef.$$

Now by construction, observe  $\triangle ABE \sim \triangle ACD$  and  $\triangle AED \sim \triangle ABC$ . (Why? Hint: Basic angle chasing theorems in the circle. Look at Figure 2.) Thus, we have  $\frac{a}{|BE|} = \frac{e}{c}$  and  $\frac{d}{|ED|} = \frac{e}{a}$ , so  $ac = e \cdot |BE|$  and  $bd = e \cdot |ED|$ .

Recall  $|BE| + |ED| = f$ . Hence, adding them gives

$$ac + bd = e \cdot |BE| + e \cdot |ED| = e(|BE| + |ED|) = ef,$$

which ends the proof. As for the degenerate case, note that quadrilateral  $ABCD$  then lies on a circle with radius  $\infty$ . Doing **Exercise 2.1** will make this clearer for you. If you're not too confident with that, you may also prove it by bashing out some lengths.  $\square$

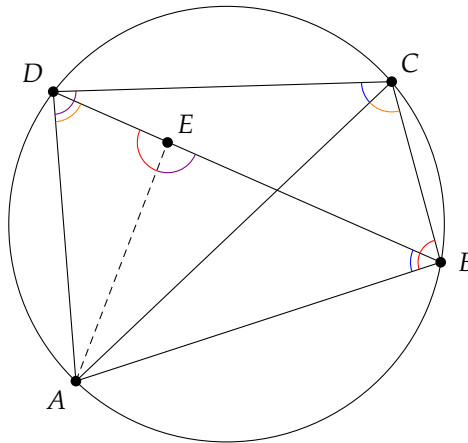


Figure 2: Angle Chasing for similar triangles

In fact, Ptolemy's Theorem is just a tiny part of what we will be looking at. You have probably heard of the *triangle inequality*. But have you ever wondered about whether there is a similar inequality for quadrilaterals? Then *Ptolemy's Inequality* is what you were looking for.

#### Ptolemy's Inequality

If  $A, B, C, D$  are four points in the plane <sup>a</sup>, then

$$|AB| \cdot |CD| + |AD| \cdot |BC| \geq |AC| \cdot |BD|.$$

Equality is achieved, if and only if  $ABCD$  is a (possibly degenerate) cyclic quadrilateral.

<sup>a</sup>Interestingly enough, that can actually be generalised for all dimensions.

*Proof.* Again, take a look by yourself first.

We want to kind of mimic the proof of the equality case. That is, our proof now will be motivated by the proof you've already seen. Let  $X$  and  $Y$  be points on the diagonals  $BD$  and  $AC$ , such that

$\angle BAX = \angle CAD$  and  $\angle YBA = \angle DCA$ . Let  $E$  be the intersection of  $AX$  and  $BY$ . Then, it is easy to see that  $\angle BAC = \angle EAD$ , so  $\triangle ABE \sim \triangle ACD$ . That gives

$$\frac{|AB|}{|AC|} = \frac{|BE|}{|CD|} \iff |AB| \cdot |CD| = |AC| \cdot |BE|. \quad (1)$$

The similarity also gives us  $\frac{|AB|}{|AC|} = \frac{|AE|}{|AD|}$ . Since  $\angle BAE = \angle EAD$ , by  $S : A : S$  similarity, we get  $\triangle AED \sim \triangle ABC$ . This similarity then gives

$$\frac{|AD|}{|AC|} = \frac{|ED|}{|BC|} \iff |AD| \cdot |BC| = |AC| \cdot |ED|. \quad (2)$$

Adding the equations (1) and (2) yields

$$|AB| \cdot |CD| + |DA| \cdot |BC| = |AC| \cdot (|BE| + |ED|) \geq |AC| \cdot |BD|$$

by the triangle inequality. Equality holds, if and only if  $\triangle BED$  is degenerate, that is  $E$  lies on  $BD$ . That happens if and only if  $\angle DCA = \angle YBA = \angle DBA$  or equivalently if  $ABCD$  is a (possibly degenerate) cyclic quadrilateral.  $\square$

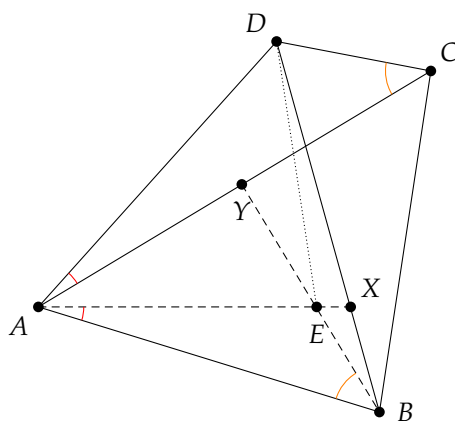


Figure 3: Mimicing the equality case with two new points

We'll proceed by discovering more proofs of the theorems. You may skip this, if you want to, but it is good practice and you should definitely do it somewhere. There are elegant proofs to Ptolemy, as well as instructive ones that explain the method being used very well.

**Exercise 2.1.** Prove Ptolemy's Inequality with *Inversion*. Try to cover the equality case with that proof as well.

**Exercise 2.2.** Prove Ptolemy's Inequality with *Complex Numbers*. Again, try to cover the equality case.

**Exercise 2.3.** Prove Ptolemy's Inequality with *Simson's Line*. Cover the equality case.

**Exercise 2.4.** Prove Ptolemy's Theorem with *trigonometry*. Think of *Addition Theorems* or the *Law of Cosine*.

**Exercise 2.5.** Can you find different proofs that weren't mentioned in this handout? <sup>1</sup>

<sup>1</sup>If you do, then feel free to send me your proof! Contact me at music.zhu@gmail.com.

### 3 Example Problems

In this section, I will be presenting 2 problems to give a general idea of how Ptolemy's Theorem may be used. In specific, I will try to explain motivational steps and include a write-up as I'd do it in a contest. You should try them yourself before reading the solution.

#### 3.1 Carnot's Theorem - Noticing Configurations

**Problem 3.1.** Let  $d_a, d_b, d_c$  be the distances from the circumcenter of an acute triangle to its sides and let  $R$  and  $r$  be its circumradius and inradius respectively. Prove that  $d_a + d_b + d_c = R + r$ .

**Solving 3.1.** It's good to start a problem labeling key objects. Let's do that. Let the feet of the perpendiculars from the circumcenter  $O$  to triangle  $ABC$  be  $X, Y, Z$  and let  $a, b, c$  be the sidelengths. Then, I see that  $XBYO, YCZO, ZAXO$  are all cyclic as they have two opposite right angles. I also note that  $X, Y, Z$  are actually midpoints of the sides. That also tells me that  $\triangle XYZ$  is actually the *Medial Triangle*. A cool configuration I have back in my mind! So  $|XY| = \frac{1}{2}b, |YZ| = \frac{1}{2}c, |ZX| = \frac{1}{2}a$ . For convenience, call  $\frac{1}{2}a = a', \frac{1}{2}b = b', \frac{1}{2}c = c'$ . Also, we defined a lot of heights with  $d_a, d_b, d_c$  onto those sides. It tells me that I should look out for areas. That technique is often used when the lengths of the perpendiculars from a point to triangle sides is given.

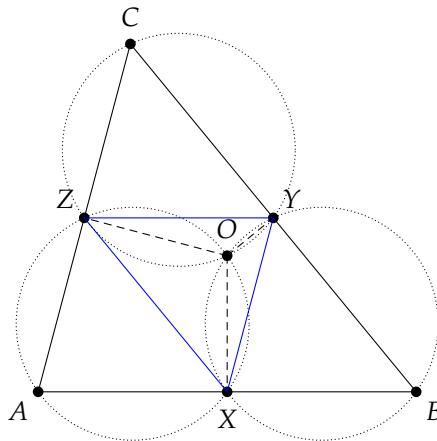


Figure 4: The Medial Triangle and all its might

Anyway, I proceed by using Ptolemy three times on the three cyclic quadrilaterals  $XBYO, YCZO, ZAXO$ , since I know the side lengths of those quadrilateral well and it is connected to the relation I am trying to prove. It gives me

$$\begin{aligned} a'd_c + c'd_a &= Rb' \\ a'd_b + b'd_a &= Rc' \\ b'd_c + c'd_b &= Ra' \end{aligned}$$

It seems like, somehow combining them could lead to some canceling. So I try some combinations. One of them being adding all three equations. That yields

$$a'(d_b + d_c) + b'(d_c + d_a) + c'(d_a + d_b) = R(a' + b' + c').$$

I'd like to divide by  $a' + b' + c'$ . I'd already have the summand  $R$  on the right hand side. And the left hand side almost has factor  $a' + b' + c'$ . So perhaps I'd add something to the equation. That is why I add  $a'd_a + b'd_b + c'd_c$  to both sides. This gives

$$(a' + b' + c')(d_a + d_b + d_c) = R(a' + b' + c') + a'd_a + b'd_b + c'd_c.$$

So now it suffices to prove

$$\frac{a'd_a + b'd_b + c'd_c}{a' + b' + c'} = r.$$

I understand the denominator very well, it is just the semiperimeter, which we call  $s$ . So it is equivalent to

$$a'd_a + b'd_b + c'd_c = rs = [ABC]$$

where  $[ABC]$  is the area of  $\triangle ABC$ . Note that  $[ABC] = rs$  is well known. But wait. The left hand side can be easily expressed as  $[ABC]$ . I see

$$a'd_a + b'd_b + c'd_c = [BCO] + [CAO] + [ABO] = [ABC].$$

So that's where the area came into play!

*Proof.* Let  $\triangle ABC$  be one such triangle and denote its side lengths with  $a, b, c$ . Call the feet of the perpendiculars from circumcenter  $O$  to the sides  $X, Y, Z$ . Then  $XBYO$  is cyclic, since opposite angles add up to  $180^\circ$  due to  $\angle BXO = \angle OYB = 90^\circ$ . Similarly,  $YCZO, ZAXO$  are also cyclic. Note that  $X, Y, Z$  are the midpoints of the sides and  $|XY| = \frac{1}{2}b, |YZ| = \frac{1}{2}c, |ZX| = \frac{1}{2}a$  as  $\triangle XYZ$  is the medial triangle of  $\triangle ABC$ . Let  $\frac{1}{2}a = a', \frac{1}{2}b = b', \frac{1}{2}c = c'$ . Then, by Ptolemy on  $XBYO, YCZO, ZAXO$ , we get

$$a'd_c + c'd_a = Rb'$$

$$a'd_b + b'd_a = Rc'$$

$$b'd_c + c'd_b = Ra'$$

Adding them and  $a'd_a + b'd_b + c'd_c$  yields

$$d_a + d_b + d_c = R + \frac{a'd_a + b'd_b + c'd_c}{a' + b' + c'}.$$

But

$$\frac{a'd_a + b'd_b + c'd_c}{a' + b' + c'} = \frac{[BCO] + [CAO] + [ABO]}{a' + b' + c'} = \frac{[ABC]}{a' + b' + c'} = r.$$

So we are done. □

### 3.2 IMO 1995, #5 - Symmetry in Hexagon

**Problem 3.2.** Let  $ABCDEF$  be a convex hexagon with  $|AB| = |BC| = |CD|$  and  $|DE| = |EF| = |FA|$ , such that  $\angle BCD = \angle EFA = 60^\circ$ . Suppose  $G$  and  $H$  are points in the interior of the hexagon such that  $\angle AGB = \angle DHE = 120^\circ$ . Prove that  $|AG| + |GB| + |GH| + |DH| + |HE| \geq |CF|$ .

**Solving 3.2.** There are certain conditions that seem like they could be difficult to handle. Hexagons can be difficult and the inequality uses a good number of different lengths, that could be disgusting. But on the other hand, there are also several conditions that seem easy to handle. I like  $60^\circ$  or  $120^\circ$  angles and a lot of lengths are equal.

Now, the first difficulty I have had with this problem is that I did not know how to draw a diagram. And things get difficult in geometry, if you cannot visualize it, so a good diagram is crucial. Therefore, the first task should be trying to draw such a hexagon. For that, I made a quick scratch without the ruler to see, if there is any special property that I could use to draw the hexagon. And with that, I noticed that  $\triangle BCD$  and  $\triangle EFA$  are actually equilateral, as they are both isosceles with a  $60^\circ$  angle. That lets me label the diagram a bit more by letting

$$|AB| = |BC| = |CD| = |BD| = a \quad \text{and} \quad |DE| = |EF| = |FA| = |EA| = b.$$

Now, I'd start by drawing segment  $AB$ . I could follow up with a segment  $BD$  with equal length. I knew that I could just construct point  $C$  on the far right, so I didn't care about that for now. I knew that



be hard to prove it rigorously. Some congruency or rotating should do it. Cool, I've solved an IMO problem. Yay.

*Proof.* Let  $C'$  and  $F'$  be points outside of hexagon  $ABCDEF$  such that  $\triangle AC'B$  and  $\triangle DF'E$  are equilateral. As  $\angle AGB + \angle BC'A = 180^\circ$ , quadrilateral  $AC'BD$  is cyclic. Therefore, by Ptolemy's Theorem we have

$$|AG| \cdot |BC'| + |GB| \cdot |C'A| = |C'G| \cdot |AB| \iff |AG| + |GB| = |C'G|,$$

since  $|BC'| = |C'A| = |AB|$  by construction. Similarly,  $|DH| + |HE| = |F'H|$ . Thus,

$$|AG| + |GB| + |GH| + |DH| + |HE| = |C'G| + |GH| + |HF'| \geq |F'C'|,$$

with two uses of the triangle inequality  $|C'G| + |GH| \geq |C'H|$  and  $|C'H| + |HF'| \geq |F'C'|$ . But we also have  $|FC| = |F'C'|$ , as hexagons  $AXBDYE$  and  $ABCDEF$  are congruent because all corresponding sides and angles are equal. □

## 4 Problems - Practice makes perfect

In the concluding section <sup>2</sup>, I will include several problems for you to practice on your own. They are roughly arranged in difficulty, even though I cannot assure that you will feel the same way. Take your time for the problems and don't give up too early.

Most problems will be solvable with Ptolemy, you might find solutions without using Ptolemy, though. I even threw in a few problems that can not necessarily be solved with Ptolemy to let you think about all possible approaches, instead of being too focused on Ptolemy. (Or maybe I didn't? Find it out by yourself!)

Also, if you know cool problems related to Ptolemy or any of the problems in this article, please contact me as well! Have fun and good luck!

**Problem 4.1** (NIMO 14.1). Let  $A, B, C, D$  be four points on a line in this order. Suppose that  $|AC| = 25$ ,  $|BD| = 40$ , and  $|AD| = 57$ . Compute  $|AB| \cdot |CD| + |AD| \cdot |BC|$ .

**Problem 4.2.** Point  $P$  is chosen on the arc  $CD$  of the circumcircle of a square  $ABCD$ . Prove that

$$|PA| + |PC| = \sqrt{2} \cdot |PB|.$$

**Problem 4.3.** Let triangle  $ABC$  be isosceles with  $|AC| = |BC|$  and let  $P$  be a point of arc  $AB$  of its circumcircle. Then prove that  $\frac{|PA| + |PB|}{|PC|}$  is constant for all possible points  $P$ .

**Problem 4.4.** The angle bisector of  $\angle BAC$  of triangle  $ABC$  meets its circumcircle at  $D$ . Prove that

$$|AB| + |AC| \leq 2 \cdot |AD|.$$

**Problem 4.5.** Let  $ABCD$  be a cyclic quadrilateral with  $\angle CBA = \angle ADC = 90^\circ$ . Prove that

$$|BD| = |AC| \cdot \sin \angle BAD.$$

**Problem 4.6** (Baltic Way 2001, #7). A parallelogram  $ABCD$  is given. A circle passing through  $A$  meets the line segments  $AB, AC$  and  $AD$  at inner points  $M, K, N$ , respectively. Prove that

$$|AB| \cdot |AM| + |AD| \cdot |AN| = |AK| \cdot |AC|.$$

<sup>2</sup>I might make documents with hints and solutions in the near future. But until that happens, feel free to ask me via music.zhu@gmail.com, if any questions come up. I'd also take a look at your solutions, if you want to.



**Problem 4.7** (Geometry Revisited, 1.9.1). Let  $Q$  be a point of segment  $BC$  of an equilateral triangle  $ABC$ . Let line  $AQ$  meet the circumcenter again at  $P$ . Then prove

$$\frac{1}{|PA|} + \frac{1}{|PC|} = \frac{1}{|PQ|}.$$

**Problem 4.8** (Germany 2014). Let  $ABCDEFGH$  be a regular heptagon with side length 1. Prove that

$$\frac{1}{|AC|} + \frac{1}{|AD|} = 1.$$

**Problem 4.9.** Let  $\alpha = \frac{\pi}{7}$ . Prove that

$$\frac{1}{\sin \alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha}.$$

**Problem 4.10** (HMMT 2013 Team - #6). Let triangle  $ABC$  satisfy  $2|BC| = |AB| + |AC|$  and have incenter  $I$  and circumcircle  $\omega$ . Let  $D$  be the intersection of  $AI$  and  $\omega$  (with  $A, D$  distinct). Prove that  $I$  is the midpoint of  $AD$ .

**Problem 4.11** (IMO 2001, #6). Let  $a > b > c > d$  be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that  $ab + cd$  is not prime.

## References

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## 5 Hints - Giving up in war

**Hint 5.1.** Recheck Ptolemy's Theorem and don't left out any details. Have you done **Exercise 2.1**?

**Hint 5.2.** Just straightforward Ptolemy.

**Hint 5.3.** Just straightforward Ptolemy.

**Hint 5.4.** Recall that  $D$  is the midpoint of arc  $AC$ .

**Hint 5.5.** To be continued...